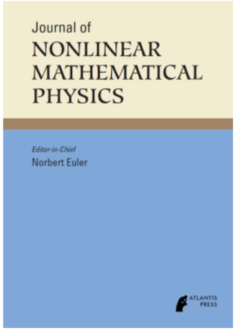

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THE INHOMOGENEOUS INVARIANCE QUANTUM GROUP OF Q-DEFORMED BOSON ALGEBRA WITH CONTINUOUS PARAMETERS

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We present a q-deformed boson algebra using continuous momentum parameters and investigate its inhomogeneous invariance quantum group.

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1. Introduction

Quantum group theory is related to integrable systems. To solve some complicated problems classical groups are not sufficient and more complex structures are used. These structures are quantum groups and they are algebraic objects [1, 2]. There are many approaches to obtain quantum groups. One way is called Drinfeld's approach [3]. In that approach by defining a deformation parameter on usual Lie algebra one can construct a quantum group. The most familiar quantum group generated from a Lie algebra is $SU_q(2)$, here q is deformation parameter and it can take values between 0 and 1.

$$T = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix},$$

the elements satisfy $ab = qba, ab^* = qb^*a, bb^* = b^*b$, $[a, a^*] = (q-1)bb^*$ and the unitarity condition is: $TT^* = T^*T = \mathbf{1}$, also the q-determinant is equal to one: $\det_q(T) = aa^* + bb^* = a^*a + qb^*b = 1$.

This procedure can be applied to some physical structure and one can get deformed physical systems. The most known example is deformed boson algebra. By defining a deformation parameter on boson algebra one can write a deformed boson algebra which is given by;

$$a_i a_j^* - q_{ij} a_j^* a_i = \delta_{ij}, \tag{1.1}$$

$$a_i a_j - a_j a_i = 0 \tag{1.2}$$

where

$$q_{ij} = (q-1)\delta_{ij} + 1. \tag{1.3}$$

Here i and j run from 1 to n. Here a and a^* are creation and annihilation operators respectively.

Another approach is realized by Woronowicz. The structures obtained are called matrix quantum groups. The idea of matrix quantum groups involves noncommutative comultiplication [4]. Also Manin has shown that one can obtain quantum groups by considering linear transformations on a quantum plane [5]. One can build up a matrix whose elements satisfy Hopf algebra axioms. In this paper we use the term "invariance quantum group" to describe a Hopf algebra such that a particle algebra forms a right module of the Hopf algebra.

Using Drinfeld's and Woronowicz's approach one can find inhomogeneous invariance quantum groups of q deformed oscillators. This quantum group is Bosonic Inhomogeneous Deformed General Linear Quantum Group, $BIGL_{q_{ij},q_{ij}^{-1}}(2d)$ [6].

In quantum field theory the operators are not discrete instead they are operators which have continuous parameters and the parameter is usually momentum which is shown by p. In this manner boson algebra can be written with continuous parameters, as;

$$a(p)a^*(p') - a^*(p')a(p) = \delta(p - p'), \tag{1.4}$$

$$a(p)a(p') - a(p')a(p) = 0. \tag{1.5}$$

Now a question arises: if a q deformed oscillator is written using continuous parameters what will be its inhomogeneous quantum invariance group? In this study we will build a q oscillator algebra with continuous parameters and find its inhomogeneous invariance quantum group.

2. $BIGL_{g,g^{-1}}(2, \mathfrak{C})$

Now we define a q oscillator algebra which the operators of the algebra have continuous parameters. The bosonic creation and annihilation operators are $a(p)$ and $a^*(p)$, here p is thought as momentum and $p \in \mathbb{R}$. The q deformed boson algebra which has continuous parameters can be written as

$$a(p)a^*(p') - g(q; p, p')a^*(p')a(p) = \delta(p - p'), \tag{2.1}$$

$$a(p)a(p') - a(p')a(p) = 0, \tag{2.2}$$

where

$$g(q;p,p') = \begin{cases} q & p = p', \\ 1 & p \neq p'. \end{cases} \tag{2.3}$$

It can be easily seen that Eqs. (2.1) and (2.2) are compatible with Eqs. (1.1) and (1.2).

Now we consider an inhomogeneous linear transformation on creation and annihilation operators of the algebra. The transformed operators can be written as;

$$a(p)' = \int \alpha(p,k) \otimes a(k)dk + \int \beta(p,k) \otimes a^*(k)dk + f(p) \otimes 1, \tag{2.4}$$

$$a^*(p)' = \int \beta^*(p,k) \otimes a(k)dk + \int \alpha^*(p,k) \otimes a^*(k)dk + f^*(p) \otimes 1. \tag{2.5}$$

The parameters $\alpha(p,k)$ and $\beta(p,k)$ are not necessarily commutative parameters and $f(p)$ is a noncommutative parameter. One can write the transformation in matrix form.

$$M = \left(\begin{array}{cc|c} \alpha & \beta & f \\ \beta^* & \alpha^* & f^* \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A & F \\ \hline 0 & 1 \end{array}\right) \tag{2.6}$$

Here A is homogeneous and F is inhomogeneous part of the transformation matrix M. We want the transformed operators satisfy the algebra which is defined in Eqs. (2.1) and (2.2). To leave the algebra invariant there should be commutation relations between the elements of transformation matrix M. The commutation relations

$$\alpha(p,k)\alpha(p',l) = \alpha(p',l)\alpha(p,k), \tag{2.7}$$

$$\alpha(p,k)\alpha^*(p',l) = \frac{g(q;p,p')}{g(q;k,l)}\alpha^*(p',l)\alpha(p,k), \tag{2.8}$$

$$\alpha(p,k)\beta(p',l) = g(q;k,l)^{-1}\beta(p',l)\alpha(p,k), \tag{2.9}$$

$$\alpha(p,k)\beta^*(p',l) = g(q;p,p')\beta^*(p',l)\alpha(p,k), \tag{2.10}$$

$$\alpha(p,k)f(p') = f(p')\alpha(p,k), \tag{2.11}$$

$$\alpha(p,k)f^*(p') = g(q;p,p')f^*(p')\alpha(p,k), \tag{2.12}$$

$$\beta(p,k)\beta(p',l) = \beta(p',l)\beta(p,k), \tag{2.13}$$

$$\beta(p,k)\beta^*(p',l) = g(q;p,p')g(q;k,l)\beta^*(p',l)\beta(p,k), \tag{2.14}$$

$$\beta(p,k)f(p') = f(p')\beta(p,k), \tag{2.15}$$

$$\beta(p,k)f^*(p') = g(q;p,p')f^*(p')\beta(p,k), \tag{2.16}$$

$$f(p)f(p') - f(p')f(p) = \int \alpha(p,k)\beta(p',k)dk - \int \alpha(p',k)\beta(p,k)dk, \tag{2.17}$$

$$f(p)f^*(p') - g(q;p,p')f^*(p')f(p) = \delta(p-p') - \int \alpha(p,k)\alpha^*(p',k)dk + g(q;p,p')\int \beta^*(p',k)\beta(p,k)dk, \tag{2.18}$$

together with their hermitean conjugates.

In order to check that the transformation is a quantum group we should find the coproduct, the counit and the coinverse of the transformation. The coproduct of the transformation matrix M can be found via tensor matrix multiplication.

$$\Delta(M) = M \dot{\otimes} M \tag{2.19}$$

this gives;

$$\Delta(\alpha(p,k)) = \int \alpha(p,\eta) \otimes \alpha(\eta,k)d\eta + \int \beta(p,\eta) \otimes \beta^*(\eta,k)d\eta, \tag{2.20}$$

$$\Delta(\beta(p,k)) = \int \alpha(p,\eta) \otimes \beta(\eta,k)d(\eta) + \int \beta(p,\eta) \otimes \alpha^*(\eta,k)d\eta, \tag{2.21}$$

$$\Delta(f(p)) = \int \alpha(p,\eta) \otimes f(\eta)d\eta + \int \beta(p,\eta) \otimes f^*(\eta)d(\eta) + f(p) \otimes 1, \tag{2.22}$$

$$\Delta(\alpha^*(p,k)) = \int \alpha^*(p,\eta) \otimes \alpha^*(\eta,k)d\eta + \int \beta^*(p,\eta) \otimes \beta(\eta,k)d\eta, \tag{2.23}$$

$$\Delta(\beta^*(p,k)) = \int \alpha^*(p,\eta) \otimes \beta^*(\eta,k)d\eta + \int \beta^*(p,\eta) \otimes \alpha(\eta,k)d\eta, \tag{2.24}$$

$$\Delta(f^*(p)) = \int \alpha^*(p,\eta) \otimes f^*(\eta)d\eta + \int \beta^*(p,\eta) \otimes f(\eta)d\eta + f^*(p) \otimes 1, \tag{2.25}$$

$$\Delta(1) = 1 \otimes 1. \tag{2.26}$$

The counit and the coinverse are defined as;

$$\varepsilon(M) = \mathcal{I}, \tag{2.27}$$

$$S(M) = M^{-1}. \tag{2.28}$$

In order to find the inverse of transformation matrix T we should find the inverse of homogeneous part of transformation matrix.

$$T^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}F \\ 0 & 1 \end{pmatrix}, \tag{2.29}$$

where

$$A = \begin{pmatrix} \alpha(p,k) & \beta(p,k) \\ \beta^*(p,k) & \alpha^*(p,k) \end{pmatrix} \tag{2.30}$$

and

$$F = \begin{pmatrix} f(p) \\ f^*(p) \end{pmatrix}. \tag{2.31}$$

The commutation relations between the elements of homogeneous part A are;

$$\alpha(p,k)\alpha(p',l) = \alpha(p',l)\alpha(p,k), \tag{2.32}$$

$$\alpha(p,k)\alpha^*(p',l) = \frac{g(q;p,p')}{g(q;k,l)}\alpha^*(p',l)\alpha(p,k), \tag{2.33}$$

$$\alpha(p,k)\beta(p',l) = g(q;k,l)^{-1}\beta(p',l)\alpha(p,k), \tag{2.34}$$

$$\alpha(p,k)\beta^*(p',l) = g(q;p,p')\beta^*(p',l)\alpha(p,k), \tag{2.35}$$

$$\beta(p,k)\beta(p',l) = \beta(p',l)\beta(p,k), \tag{2.36}$$

$$\beta(p,k)\beta^*(p',l) = g(q;p,p')g(q;k,l)\beta^*(p',l)\beta(p,k). \tag{2.37}$$

We should remember that all $g(q;p,p')$'s are either q or 1. Equations (2.32)–(2.37) have two parameter deformed quantum group $GL_{p,q}$ [7] structure. One can write the inverse of A in light of Schirrmacher's $GL_{p,q}$. In our case $p = g(q;k,l)^{-1}$ and $q = g(q;p,p')^{-1}$.

Since the elements of matrix T have coproduct, counit and coinverse we can say that this transformation is a quantum group. This group called as $BIGL_{g,g^{-1}}(2,\mathfrak{C})$ Bosonic Inhomogeneous Deformed General Linear Quantum Group with continuous parameter.

3. Conclusion

In this study we have shown that there is an inhomogeneous invariance quantum group for deformed boson algebra with continuous parameters. This quantum group is $BIGL_{g,g^{-1}}(2)$. It is easy to see that the inhomogeneous part has bosonic structure.

The homogeneous part of $BIGL_{g,g^{-1}}(2,\mathfrak{C})$ has also a quantum group structure. This is $GL_{g,g^{-1}}(2)$. If one can choose $g = 1$ the quantum group becomes $BISp(2,\mathfrak{C})$, Bosonic Inhomogeneous Symplectic Quantum Group with continuous parameter [8].

In field theory a field $\varphi(x)$ can be written using creation and annihilation operators. Using the deformed creation and annihilation operators $a(p)$ and $a^*(p)$ one can construct a deformed field theory. Also using transformed creation and annihilation operators one can find a quantum group for deformed quantum field theory. The theory of quantum groups can be helpful in the generalization of quantization and a more consistent approach to interacting field theories.

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