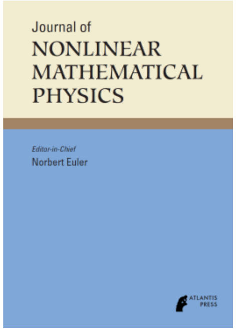

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ON (q, h)-ANALOGUE OF FRACTIONAL CALCULUS

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The paper discusses fractional integrals and derivatives appearing in the so-called (q, h)-calculus which is reduced for $h = 0$ to quantum calculus and for $q = h = 1$ to difference calculus. We introduce delta as well as nabla version of these notions and present their basic properties. Furthermore, we give comparisons with the known results and discuss possible extensions to more general settings.

Keywords: Fractional integral; fractional derivative; (q, h)-calculus; time scale; generalized Gamma function.

1. Introduction

In last decades, fractional (or non-integer) differentiation has played an important role in various areas ranging from mechanics to image processing. The essentials of the corresponding mathematical theory were discovered over 300 years ago and since its origin many outstanding mathematicians have contributed to this field. Their fundamental results have been surveyed, e.g. in the monographs [15] and [18].

The study of fractional calculus in discrete settings has been initiated in [1], [7] and [2]. While the papers [1] and [2] present the introduction to fractional q-derivatives and q-integrals, the paper [7] discusses basics of fractional difference calculus. Among other significant papers dealing with these problems we can mention, e.g. [3], [4] or [16], where discrete analogues of some topics of continuous fractional calculus (Laplace transform, Mittag–Leffler function) have been developed. Although many interesting results have been achieved, the theory of discrete fractional calculus remains, in general, much less developed than its continuous counterpart.

In this paper, we formulate some new results on quantum and difference fractional calculus. However, the main objective of this paper does not consist in their separate investigation, but especially in the unification and generalization of some notions and results achieved in the framework of these discrete settings. Both aspects — unification and generalization — are typical features of the theory of time scales introduced by S. Hilger with the view of joint investigation of continuous and discrete analysis. Although this theory forms a rapidly growing area of research with many applications (see, e.g. [6]), the discussions on the fractional calculus on time scales are at the beginning. In this connection we refer to papers [3] and [4] which can be taken for links between conventional discrete fractional calculus and the corresponding time scale calculus. Although considered on concrete discrete settings, these papers contain several remarks and suggestions indicating possible extensions to more general settings.

This paper focuses on essentials of fractional calculus on a special discrete time scale forming the background for the so-called (q, h)-calculus, which can be reduced to the quantum calculus (the case $h = 0$) or to the difference calculus (the case $q = h = 1$). Note that (q, h)-analysis of many important mathematical topics is now becoming very urgent: we can mention, e.g. the papers [5] and [8], where various (q, h)-generalizations of classical notions and properties (such as non-commutative (q, h)-binomial formula, (q, h)-analogues of important special functions and other related questions) have been studied.

The paper is structured as follows: In Sec. 2, we recall some necessary notions and properties of the time scales theory with an emphasis to quantum and difference calculus. Section 3 deals with the introduction of (q, h)-integrals and derivatives. In Secs. 4 and 5, we extend these notions to integrals and derivatives of non-integer orders. Section 6 presents some basic properties of fractional (q, h)-integrals and derivatives. Comparisons with other relevant definitions, properties and results as well as outlines of possible extensions to more general settings are discussed in Sec. 7.

2. Preliminaries

By a time scale $\mathbb{T}$ we understand any nonempty and closed subset of real numbers with the ordering inherited from reals. For any $t \in \mathbb{T}$, we define the forward (backward) jump operator by the relation $\sigma(t) := \inf\{s \in \mathbb{T},\, s > t\}$ ($\rho(t) := \sup\{s \in \mathbb{T},\, s < t\}$) and the forward (backward) graininess function $\mu(t) := \sigma(t) - t$ ($\nu(t) := t - \rho(t)$), respectively.

The symbol $f^\Delta(t)$ is the delta derivative of $f : \mathbb{T} \to \mathbb{R}$ at $t \in \mathbb{T}$. It is defined by

$$f^\Delta(t) := \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

In particular, if $\mathbb{T} = \mathbb{R}$, then $\mu = 0$ and $f^\Delta = f'$. Considering discrete time scales (i.e. such that $\mu(t) \neq 0$ for $t \in \mathbb{T}$) $f^\Delta(t)$ exists for all $t \in \mathbb{T}$ and it is given by

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \tag{2.1}$$

Similarly we can discuss the notion of nabla derivative $f^\nabla(t)$. We recall that if $\nu(t) \neq 0$ for $t \in \mathbb{T}$, then

$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}\,. \tag{2.2}$$

The most significant discrete time scales are those originating from arithmetic and geometric sequence of real numbers, namely

$$\mathbb{T}_h^{t_0} := \{t_0 + hk,\, k \in \mathbb{Z}\}, \quad h > 0 \quad \text{and} \quad \mathbb{T}_q^{t_0} := \{t_0 q^k, k \in \mathbb{Z}\} \cup \{0\}, \;\; q > 1,$$

respectively, where $t_0 \in \mathbb{R}$. These sets form the basis for the study of h-calculus and q-calculus, in the frame of which the investigation of fractional derivatives and integrals has been already started and developed.

Note that the standard definitions of delta h-derivative and delta q-derivative of f coincide with the general formula (2.1) via the choice $\sigma(t) = t + h$ and $\sigma(t) = qt$ (provided $t_0 > 0$), respectively. Similarly, (2.2) becomes the nabla h-derivative and nabla q-derivative provided $\rho(t) = t - h$ and $\rho(t) = t/q$ (provided $t_0 > 0$), respectively.

The delta integral of f over the time scale interval $[a, b] := \{t \in \mathbb{T},\ a \le t \le b\}$, $a, b \in \mathbb{T}$ is defined by $\int_a^b f(t)\Delta t := F(b) - F(a)$, where F is an antiderivative of f, i.e. the function satisfying $F^\Delta = f$ on $\mathbb{T}$. If $a, b \in \mathbb{T}$ and $a > b$, then $\int_a^b f(t)\Delta t := -\int_b^a f(t)\Delta t$ and we put $\int_a^a f(t)\Delta t := 0$.

It is known that considering discrete time scales this delta integral exists and can be calculated (provided $a < b$) via the formula

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} f(t)\mu(t). \tag{2.3}$$

Analogously we can discuss the nabla integral of f over $[a, b]$. Note only that the relation (2.3) has to be modified as

$$\int_a^b f(t)\nabla t = \sum_{t\in(a,b]} f(t)\nu(t). \tag{2.4}$$

Both formulae (2.3) and (2.4) then imply the form of the corresponding delta and nabla h-integrals ($\mu(t) = \nu(t) = h$) or q-integrals ($\mu(t) = (q-1)t$, $\nu(t) = (1-q^{-1})t$, if $t_0 > 0$).

Now we mention an elementary overview of some necessary notions of q-calculus and h-calculus. For any real number α and any $q > 0$, $q \neq 1$ we set $[\alpha]_q := \frac{q^\alpha - 1}{q-1}$. Then we have the q-analogy of $n!$ in the form $[n]_q! := [n]_q.[n-1]_q \cdots [1]_q$ for $n = 1, 2, \ldots$, whereas for $n = 0$ we put $[0]_q! := 1$. If $q = 1$, then $[\alpha]_1 := \alpha$ and $[n]_1!$ becomes the standard factorial. Further, the q-binomial coefficient is introduced by use of relations

$$\begin{bmatrix}\alpha\\0\end{bmatrix}_q := 1, \qquad \begin{bmatrix}\alpha\\n\end{bmatrix}_q := \frac{[\alpha]_q[\alpha-1]_q\cdots[\alpha-n+1]_q}{[n]_q!},$$

where $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. The extension of the q-binomial coefficient to non-integer value n is allowed via the Γ_q function defined for $0 < q < 1$ (see [12]) as

$$\Gamma_q(t) := \frac{(q,q)_\infty(1-q)^{1-t}}{(q^t,q)_\infty},$$

where $(a,q)_\infty = \prod_{j=0}^\infty (1 - aq^j)$ and $t \in \mathbb{R}\backslash\{0, -1, -2, \ldots\}$. For $q > 1$ (see [13]) we have

$$\Gamma_q(t) := \frac{q^{\binom{t}{2}}(q^{-1},q^{-1})_\infty(q-1)^{1-t}}{(q^{-t},q^{-1})_\infty}.$$

Both functions are related by $\Gamma_q(t) = \Gamma_{1/q}(t)\, q^{\binom{t-1}{2}}$. If $q = 1$, then Γ_q is reduced to the Euler Gamma function Γ. It is easy to check that Γ_q satisfies the functional relation

$\Gamma_q(t+1) = [t]_q\,\Gamma_q(t)$. Using this property we can define

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_q := \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha-\beta+1)}, \quad \alpha,\beta,\alpha-\beta \in \mathbb{R}\setminus\mathbb{Z}^-. \tag{2.5}$$

The q-analogue of the power function is introduced as

$$(t-s)_q^{(\alpha)} := t^\alpha \frac{(s/t,q)_\infty}{(q^\alpha s/t,q)_\infty}, \quad t\neq 0,\ \ 0<q<1,\ \ \alpha\in\mathbb{R}. \tag{2.6}$$

Considering positive integers n, the expression is reduced to

$$(t-s)_q^{(n)} = t^n \prod_{j=0}^{n-1}(1-q^j s/t). \tag{2.7}$$

The h-generalization of the Gamma function is introduced in [9] by the formula

$$\Gamma_h(t) := \lim_{n\to\infty} \frac{n!\,h^n(nh)^{\frac{t}{h}-1}}{t(t+h)\cdots(t+(n-1)h)}, \quad t\in\mathbb{R}\backslash h\mathbb{Z}^-,\ \ t\neq 0,$$

where $h\mathbb{Z}^- := \{kh, k\in\mathbb{Z}^-\}$. The main property characterizing the function Γ_h is the relation $\Gamma_h(t+h) = t\,\Gamma_h(t)$ (we note that other generalizations of the Gamma function, especially with respect to (q,h)-generalizations, are the subject matter of the paper [8]). Then the h-analogue of the power function is defined by

$$t_h^{(\alpha)} := \frac{\Gamma_h(t+\alpha h)}{\Gamma_h(t)}, \quad \alpha\in\mathbb{R}, \tag{2.8}$$

which for positive integers n becomes h-factorial polynomial

$$t_h^{(n)} = \prod_{j=0}^{n-1}(t+jh). \tag{2.9}$$

As it is customary, if $t\in h\mathbb{Z}^-\cup\{0\}$ and α is not an integer, then $t_h^{(\alpha)}$ is assumed to be zero.

The final goal of this section is to propose the (q,h)-analogue of the power function which will turn out to be of great importance in next investigations. For $\alpha\in\mathbb{R}$ we define

$$(t-s)_{(q,h)}^{(\alpha)} := \begin{cases} (\tilde{t}-\tilde{s})_q^{(\alpha)} & \text{for } 0<q<1,\ \ h\geq 0,\\ (t-s)_h^{(\alpha)} & \text{for } q=1,\ \ h>0,\end{cases}$$

where $\tilde{t} := t + hq/(1-q)$ and $\tilde{s} := s + hq/(1-q)$. In other words, using (2.6) and (2.8) we have

$$(t-s)_{(q,h)}^{(\alpha)} = \begin{cases} \tilde{t}^\alpha \dfrac{\prod_{j=0}^\infty(1-q^j\tilde{s}/\tilde{t})}{\prod_{j=0}^\infty(1-q^{\alpha+j}\tilde{s}/\tilde{t})} & \text{for } 0<q<1,\ \ h\geq 0,\\ \dfrac{\Gamma_h(t-s+\alpha h)}{\Gamma_h(t-s)} & \text{for } q=1,\ \ h>0.\end{cases}$$

3. Basics of (q,h)-Derivatives and Integrals

In this section, we first introduce the two-parameter time scale $\mathbb{T}^{t_0}_{(q,h)}$ generalizing time scales $\mathbb{T}^{t_0}_h$ and $\mathbb{T}^{t_0}_q$ introduced in the previous part. Both time scales are characterized by linearity of the forward (as well as the backward) jump operator, because $\sigma(t) = t + h$ or $\sigma(t) = qt$. Then the natural unification and extension of these discrete settings is enabled by the time scale with the forward jump operator $\sigma(t) = qt + h$. To describe explicitly such a time scale we can observe that

$$\sigma^k(t) = q^k t + [k]_q h \quad \text{and} \quad \rho^k(t) = q^{-k}(t - [k]_q h) = q^{-k}t + [-k]_q h, \;\; k \in \mathbb{Z}^+,$$

where the symbols σ^k and ρ^k means the k-th iterate of σ and ρ, respectively. Then, for a given $t_0 \in \mathbb{R}^+$, we define

$$\mathbb{T}^{t_0}_{(q,h)} := \{t_0 q^k + [k]_q h, k \in \mathbb{Z}\} \cup \left\{\frac{h}{1-q}\right\}, \quad q \geq 1, \;\; h \geq 0, \;\; q + h > 1.$$

Of course, $\mathbb{T}^{t_0}_{(q,h)} = \mathbb{T}^{t_0}_q$ provided $h = 0$ and $\mathbb{T}^{t_0}_{(q,h)} = \mathbb{T}^{t_0}_h$ provided $q = 1$ (in this case we put $h/(1-q) := -\infty$). We note that the time scale $\mathbb{T}^{t_0}_{(q,h)}$ with $t_0 \in \mathbb{R}^-$ can be considered quite analogously. For the sake of definiteness we restrict our considerations to $t_0 > 0$.

The introduction of (q,h)-derivative of $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$ now follows naturally from formulae (2.1) and (2.2): Let $t \in \mathbb{T}^{t_0}_{(q,h)}$. Then we define delta and nabla (q,h)-derivative of f at t by

$$(\Delta_{(q,h)} f)(t) = \frac{f(qt+h) - f(t)}{(q-1)t + h} \quad \text{and} \quad (\nabla_{(q,h)} f)(t) = \frac{f(t) - f(q^{-1}(t-h))}{q^{-1}((q-1)t+h)}.$$

Using formulae (2.3) and (2.4) we can introduce several types of (q,h)-integrals. We employ here the following two types which enable us to present some comparisons with known definitions and results:

Let $t \in \mathbb{T}^{t_0}_{(q,h)}$ and let $t > t_0$, i.e. there exists $n \in \mathbb{Z}^+$ such that $t = t_0 q^n + [n]_q h$. Then we define delta (q,h)-integral

$$(\Delta^{-1}_{(q,h)} f)(t) := \int_{t_0}^{t} f(\tau)\Delta\tau = ((q-1)t_0 + h) \sum_{k=0}^{n-1} q^k f(t_0 q^k + [k]_q h).$$

If we consider the time scale $\mathbb{T}^{t_0}_{(q,h)}$ with t_0 replaced by t, then the integration over $[\frac{h}{1-q}, t]$ yields the definition of the nabla (q,h)-integral in the form

$$(\nabla^{-1}_{(q,h)} f)(t) := \int_{h/(1-q)}^{t} f(\tau)\nabla\tau = ((1-q^{-1})t + q^{-1}h) \sum_{k=0}^{\infty} q^{-k} f(q^{-k}t + [-k]_q h) \quad (3.1)$$

provided the infinite series converges. More generally, we can consider nabla (q,h)-integral with the lower limit $h/(1-q)$ replaced by $a \in \mathbb{T}^{t}_{(q,h)}$, $h/(1-q) \leq a < t$. If $h/(1-q) < a$, then the infinite series occuring in (3.1) is reduced to the corresponding finite series. These definitions extend standard integral introductions: in particular, if $h = 0$, then (3.1) becomes the Jackson q-integral.

Last we note that an alternative approach allowing an extension of h-derivatives and q-derivatives provides the theory of deformed derivative operators introduced and investigated in [17]. In particular, our delta (q,h)-derivative can be considered as the special case of such a deformed derivative. Possible extension to more general settings (especially with respect to fractional differentiation and integration) is a matter of further investigation.

4. Fractional (q,h)-Integrals

The aim of this section is to extend definitions of (q,h)-integrals to integrals of non-integer orders. It is well known that the Cauchy formula plays the key role in generalizations of this type. We present here its extended form generalizing the corresponding standard difference formula and q-formula. Although we consider the concrete time scale $\mathbb{T}_{(q,h)}^{t_0}$ in our next considerations, we shall use the general time scale notation (e.g. $\sigma(t)$ instead of $qt+h$, etc.) to emphasize possible extensions of some results to general time scales.

Note that the intended (q,h)-generalization can be allowed also if the modified jump operators $\hat{\sigma}(t) = q(t+h)$ and $\hat{\rho}(t) = q^{-1}t - h$ are considered. This choice generates a time scale $\hat{\mathbb{T}}_{(q,h)}^{t_0}$ slightly different from $\mathbb{T}_{(q,h)}^{t_0}$. However, we can formally rewrite the operator $\hat{\sigma}$ to the form $\hat{\sigma}(t) = qt + qh = qt + \tilde{h}$ (then of course $\hat{\rho}(t) = q^{-1}(t-\tilde{h})$). In this sense, the resulting properties and definitions are equivalent.

Theorem 1 (Nabla (q,h)-Cauchy formula). *Let $n \in \mathbb{Z}^+$, $f : \mathbb{T}_{(q,h)}^{t_0} \to \mathbb{R}$ and $a, t \in \mathbb{T}_{(q,h)}^{t_0}$. Then*

$$\begin{aligned}(I_a^n f)(t) &\equiv \int_a^t \left(\int_a^{\tau_n} \cdots \left(\int_a^{\tau_2} f(\tau_1)\nabla\tau_1\right)\cdots\nabla\tau_{n-1}\right)\nabla\tau_n \\ &= \int_a^t \frac{\prod_{j=1}^{n-1}(t-\rho^j(\tau))}{[n-1]_q!}\, q^{\binom{n-1}{2}} f(\tau)\nabla\tau. \qquad (4.1)\end{aligned}$$

Proof. This assertion can be proved by induction. Let $n \geq 2$ and assume that (4.1) holds with n replaced by $n-1$, i.e.

$$(I_a^{n-1} f)(t) = \int_a^t \frac{\prod_{j=1}^{n-2}(t-\rho^j(\tau))}{[n-2]_q!}\, q^{\binom{n-2}{2}} f(\tau)\nabla\tau. \qquad (4.2)$$

We show that (4.1) is true. Applying the nabla derivative (with respect to t) to both sides of (4.1) we get

$$(I_a^{n-1} f)(t) = \int_a^t \frac{\nabla_{(q,h)}\prod_{j=1}^{n-1}(t-\rho^j(\tau))}{[n-1]_q!}\, q^{\binom{n-1}{2}} f(\tau)\nabla\tau \qquad (4.3)$$

by use of the relation

$$\nabla_{(q,h)} \int_a^t g(t,s)\nabla s = \int_a^t \nabla_{(q,h)} g(t,s)\nabla s + g(\rho(t),t) \qquad (4.4)$$

and the property $\prod_{j=1}^{n-1}(\rho(t)-\rho^j(t))/[n-1]_q! = 0$. To calculate the right-hand side of (4.3) we write

$$\begin{aligned}
\nabla_{(q,h)}\prod_{j=1}^{n-1}(t-\rho^j(\tau)) &= \frac{\prod_{j=1}^{n-1}(t-\rho^j(\tau))-\prod_{j=1}^{n-1}(\rho(t)-\rho^j(\tau))}{\nu(t)}\\
&= \frac{\prod_{j=1}^{n-1}(t-q^{-j}(\tau-[j]_q h))-\prod_{j=1}^{n-1}(q^{-1}(t-h)-q^{-j}(\tau-[j]_q h))}{q^{-1}((q-1)t+h)}\\
&= \frac{\prod_{j=1}^{n-1}(t-q^{-j}(\tau-[j]_q h))-q^{1-n}\prod_{j=1}^{n-1}(t-h-q^{1-j}(\tau-[j]_q h))}{q^{-1}((q-1)t+h)}\\
&= \frac{\prod_{j=1}^{n-1}(t-q^{-j}(\tau-[j]_q h))-q^{1-n}\prod_{j=1}^{n-1}(t-q^{1-j}(\tau-[j-1]_q h))}{q^{-1}((q-1)t+h)}\\
&= \frac{\prod_{j=1}^{n-2}(t-q^{-j}(\tau-[j]_q h))(t-q^{1-n}(\tau-h[n-1]_q)-q^{1-n}(t-\tau))}{q^{-1}((q-1)t+h)}\\
&= \prod_{j=1}^{n-2}(t-q^{-j}(\tau-[j]_q h))q^{2-n}\frac{(q^{n-1}-1)t+h[n-1]_q}{(q-1)t+h}\\
&= \prod_{j=1}^{n-2}(t-q^{-j}(\tau-[j]_q h))q^{2-n}[n-1]_q.
\end{aligned}$$

Hence

$$\frac{\nabla_{(q,h)}\prod_{j=1}^{n-1}(t-\rho^j(\tau))}{[n-1]_q!}q^{\binom{n-1}{2}}f(\tau) = \frac{\prod_{j=1}^{n-2}(t-\rho^j(\tau))}{[n-2]_q!}q^{\binom{n-2}{2}}f(\tau)$$

and (4.3) becomes (4.2). Since the antiderivative of zero is a constant function, both sides of (4.1) differ at most identically by a constant C. If we substitute any $t\in\mathbb{T}_{(q,h)}^{t_0}$, then $C=0$ and the nabla (q,h)-Cauchy formula is proved. $\square$

Theorem 2 (Delta (q,h)-Cauchy formula). *Let $n\in\mathbb{Z}^+$, $f:\mathbb{T}_{(q,h)}^{t_0}\to\mathbb{R}$ and $a,t\in\mathbb{T}_{(q,h)}^{t_0}$. Then*

$$\begin{aligned}
(J_a^n f)(t) &\equiv \int_a^t\left(\int_a^{\tau_n}\cdots\left(\int_a^{\tau_2}f(\tau_1)\,\Delta\tau_1\right)\cdots\Delta\tau_{n-1}\right)\Delta\tau_n\\
&= \int_a^{\rho^{n-1}(t)}\frac{\prod_{j=1}^{n-1}(t-\sigma^j(\tau))}{[n-1]_q!}f(\tau)\,\Delta\tau. \qquad (4.5)
\end{aligned}$$

Proof. The delta (q,h)-Cauchy formula can be proved similarly as in the previous case. Note only that the property

$$\Delta_{(q,h)}\int_a^{\rho^{n-1}(t)}g(t,s)\Delta s = \int_a^{\rho^{n-2}(t)}\Delta_{(q,h)}g(t,s)\Delta s + g(t,\rho^{n-1}(t))\frac{\mu(\rho^{n-1}(t))}{\mu(t)}$$

instead of (4.4) has to be employed now. $\square$

Remark 1. It is easy to check that the n-th multiple integral $(I_a^n f)(t)$ vanishes for all $\rho^{n-1}(a) \le t \le a$ and $(J_a^n f)(t)$ vanishes for all $a \le t \le \sigma^{n-1}(a)$.

Remark 2. The term $\prod_{j=1}^{n-1}(t-\rho^j(\tau))$ occuring in the nabla (q,h)-Cauchy formula can be taken for the extension of the Pochhammer symbol for shifting factorials to more general settings. This extension can be characterized by the relation

$$\nabla_{(q,h)} \prod_{j=1}^{n}(t-\rho^j(\tau)) = [n]_{q^{-1}} \prod_{j=1}^{n-1}(t-\rho^j(\tau)) \tag{4.6}$$

which generalizes the key derivative properties of h-calculus and q-calculus, namely

$$\nabla_h \prod_{j=1}^{n}(t+jh) = n \prod_{j=1}^{n-1}(t+jh) \quad \text{and} \quad \nabla_q \prod_{j=1}^{n}(t-q^{-j}\tau) = [n]_{q^{-1}} \prod_{j=1}^{n-1}(t-q^{-j}\tau),$$

respectively. Recall that the property (4.6) has been employed in the proof of Theorem 1 (see the calculations on the right-hand side of (4.3)).

A different way of (q,h)-extension of these symbolic powers provides the paper [8]. The suggested generalization originates from the relations (2.7) and (2.9) which are then extended in [8, Definitions 4 and 5] to (q,h)-Pochhammer symbol or (q,h)-power defined by

$$[t]_{(q,h)}^{(n)} := \prod_{j=0}^{n-1}[t+jh]_q, \quad (t-s)_{(q,h)}^{(n)} := \prod_{j=0}^{n-1}(t-q^{jh}s), \quad 0<q<1. \tag{4.7}$$

We note that these extensions of symbolic powers are quite straightforward and different from our introduction: in particular, comparing with our alternative approach we can see that the property (4.6) is no longer valid provided $\prod_{j=1}^{n}(t-\rho^j(\tau))$ is replaced by (q,h)-powers from (4.7).

Another related topic discussed in the framework of (q,h)-analysis is introduced in [5]. The main idea consists in a generalization of the q-binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k x^{n-k} \tag{4.8}$$

obtained by use of a specific non-commutative rule $xy = qyx$. Considering the second parameter h and using the rule $xy = qyx + hy^2$, the paper [5] presents the (q,h)-extension of (4.8) involving the (q,h)-binomial coefficient defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} := \begin{bmatrix} n \\ k \end{bmatrix}_q h^k \prod_{j=0}^{k-1}(h^{-1}+[j]_q) \tag{4.9}$$

(for the other extension of (q,h)-binomial formula we refer to [19]). In general, comparing this generalization of the q-case with our investigations on (q,h)-fractional calculus, we emphasize that both matters discuss different types of (q,h)-extensions. In particular, the (q,h)-binomial coefficient is not necessary in our considerations and its "natural" introduction in the framework of (q,h)-calculus remains questionable. For instance, analogously to the relation (2.5), we can formally define the (q, h)-binomial coefficient by use of the $\Gamma_{(q,h)}$ function (introduced in [8]) to find that this procedure leads to a symbol different from (4.9). Similarly, it is possible to rewrite the (q, h)-binomial coefficient from (4.9) as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} = \frac{\prod_{j=1}^{n}(1-\sigma^j(1))}{\prod_{j=1}^{n-k}(1-\sigma^j(1))\prod_{j=1}^{k}(1-\sigma^j(1))}\prod_{j=0}^{k-1}(1+\sigma^j(0))$$

(here we understand that σ can operate on any needed real $t \geq 0$). However, the obtained formula seems to have no relation to the symbols utilized in (q, h)-fractional calculus.

Remark 3. Since $[n-1]_q! = q^{\binom{n-1}{2}}[n-1]_{q^{-1}}!$ and

$$\begin{aligned}
\prod_{j=1}^{n-1}(t-\rho^j(\tau)) &= \prod_{j=1}^{n-1}\left(t - q^{-j}\tau - \frac{q^{-j}-1}{q-1}h\right) \\
&= \prod_{j=1}^{n-1}\left(t + \frac{h}{q-1} - q^{1-j}\left(q^{-1}\tau + \frac{q^{-1}-1}{q-1}h + \frac{h}{q-1}\right)\right) \\
&= \prod_{j=1}^{n-1}\left(t + \frac{hq^{-1}}{1-q^{-1}} - q^{1-j}\left(q^{-1}\tau + [-1]_q h + \frac{hq^{-1}}{1-q^{-1}}\right)\right) \\
&= \prod_{j=1}^{n-1}(\tilde{t} - q^{1-j}\widetilde{\rho(\tau)}) = \tilde{t}^{n-1}\prod_{j=0}^{n-2}(1-q^{-j}\widetilde{\rho(\tau)}/\tilde{t}) \\
&= \tilde{t}^{n-1}\frac{(\widetilde{\rho(\tau)}/\tilde{t}, q^{-1})_\infty}{(q^{1-n}\widetilde{\rho(\tau)}/\tilde{t}, q^{-1})_\infty} = (t-\rho(\tau))^{(n-1)}_{(q^{-1},h)}
\end{aligned}$$

(here we assume that $q > 1$; the case $q = 1$ is trivial), we can rewrite (4.1) as

$$(I_a^n f)(t) = \frac{1}{[n-1]_{q^{-1}}!}\int_a^t (t-\rho(\tau))^{(n-1)}_{(q^{-1},h)} f(\tau)\nabla\tau.$$

Analogously we can obtain the delta Cauchy formula (4.5) in the form

$$(J_a^n f)(t) = \frac{1}{[n-1]_q!}\int_a^{\rho^{n-1}(t)} (t-\sigma^{n-1}(\tau))^{(n-1)}_{(q^{-1},h)} f(\tau)\Delta\tau.$$

Now we are in a position to define the nabla fractional integral.

Definition 1 (Nabla (q, h)-fractional integral). Let $\alpha \in \mathbb{R}^+$, $f : \mathbb{T}_{(q,h)}^{t_0} \to \mathbb{R}$ and let $a, t \in \mathbb{T}_{(q,h)}^{t_0}$. Then we define the nabla (q, h)-fractional integral of f at t by

$$({}_a\nabla_{(q,h)}^{-\alpha} f)(t) := \frac{1}{\Gamma_{q^{-1}}(\alpha)}\int_a^t (t-\rho(\tau))^{(\alpha-1)}_{(q^{-1},h)} f(\tau)\nabla\tau. \tag{4.10}$$

The next assertion presents another expression of the nabla (q, h)-fractional integral which will be utilized in Sec. 6. It can be also considered as the alternative introduction of this type of integral.

For $t_1,\, t_2 \in \mathbb{T}^{t_0}_{(q,h)}$, we define the integer $k = k(t_1, t_2)$ by the relation $t_1 = \sigma^k(t_2)$. Note that if $t_1 < t_2$, then $k < 0$ (in this case the symbol σ^k means the $-k$-th iteration of the inverse $\sigma^{-1} = \rho$). Then we have

Lemma 1. *Let $\alpha \in \mathbb{R}^+$, $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$ and let $a,\, t \in \mathbb{T}^{t_0}_{(q,h)}$. Then*

$$({}_a\nabla^{-\alpha}_{(q,h)} f)(t) = \int_a^t (\nu(\tau))^{\alpha-1} \begin{bmatrix} \alpha - 1 + k(t,\tau) \\ \alpha - 1 \end{bmatrix}_q f(\tau)\nabla\tau. \tag{4.11}$$

Remark 4. It requires only routine calculations to verify that $k(t,\tau) = \log_q \frac{\nu(t)}{\nu(\tau)}$ provided $q > 1$, $h \ge 0$ and $k(t,\tau) = (t-\tau)/h$ provided $q = 1$, $h > 0$.

Proof. Let $q > 1$, $h \ge 0$ and let $k \in \mathbb{Z}$ be such that $t = \sigma^k(\tau) = q^k\tau + [k]_q h$ which implies $\nu(t) = q^k\nu(\tau)$. If $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, then

$$\begin{aligned}
&\frac{1}{\Gamma_{q^{-1}}(\alpha)} (t - \rho(\tau))^{(\alpha-1)}_{(q^{-1},h)} \\
&\quad = \frac{q^{\binom{\alpha}{2}}}{\Gamma_q(\alpha)} (q-1)^{1-\alpha} (\nu(t))^{\alpha-1} \frac{(q^{-k-1}, q^{-1})_\infty}{(q^{-k-\alpha}, q^{-1})_\infty} \\
&\quad = \frac{q^{\binom{\alpha+k}{2} - \binom{k+1}{2}}}{\Gamma_q(\alpha)} (q-1)^{1-\alpha} (\nu(\tau))^{\alpha-1} \frac{(q^{-k-1}, q^{-1})_\infty}{(q^{-k-\alpha}, q^{-1})_\infty} \\
&\quad = (\nu(\tau))^{\alpha-1} \frac{\Gamma_q(\alpha+k)}{\Gamma_q(\alpha)\Gamma_q(k+1)} = (\nu(\tau))^{\alpha-1} \begin{bmatrix} \alpha - 1 + \log_q \frac{\nu(t)}{\nu(\tau)} \\ \alpha - 1 \end{bmatrix}_q .
\end{aligned}$$

If $\alpha \in \mathbb{Z}^+$, then

$$\begin{aligned}
&\frac{1}{\Gamma_{q^{-1}}(\alpha)} (t - \rho(\tau))^{(\alpha-1)}_{(q^{-1},h)} \\
&\quad = \frac{q^{\binom{\alpha}{2}}}{[\alpha-1]_q!} (q-1)^{1-\alpha} (\nu(t))^{\alpha-1} \prod_{j=1}^{\alpha-1} (1 - q^{-k-j}) \\
&\quad = (\nu(\tau))^{\alpha-1} \frac{[\alpha+k-1]_q [\alpha+k-2]_q \cdots [k+1]_q}{[\alpha-1]_q!} = (\nu(\tau))^{\alpha-1} \begin{bmatrix} \alpha - 1 + \log_q \frac{\nu(t)}{\nu(\tau)} \\ \alpha - 1 \end{bmatrix}_q .
\end{aligned}$$

The case $q = 1$, $h > 0$ follows from the property $\Gamma_h(t) = h^{\frac{t}{h}-1}\Gamma(\frac{t}{h})$. □

Remark 5. In the previous proof we have used the convention $1/\Gamma_q(k+1) = 0$ for $k \in \mathbb{Z}^-$. Lemma 1 then particularly implies that $({}_a\nabla^{-\alpha}_{(q,h)} f)(t) = 0$ for all $t \le a$ and $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$. If $\alpha \in \mathbb{Z}^+$, then $({}_a\nabla^{-\alpha}_{(q,h)} f)(t) = 0$ for $t = \rho^{\alpha-1}(a), \dots, \rho(a), a$.

Now we come to the introduction of the (q,h)-delta fractional integral. At first glance, the procedure leading to the introduction of this integral might be identical with the nabla case. This is, in general, true, but before adapting this procedure we must dispose with the issues concerning the upper bound $\rho^{\alpha-1}(t)$ of the corresponding integral. In particular, considering α with non-integer values we get a non-integer iteration of a given function ρ which need not belong to a given time scale.

The problem of non-integer iterations (in general principle very close to the discussed problem of fractional calculus) is completely solved in the frame of continuous iterations theory (see, e.g. [20]). Here we only outline the standard procedure leading to the desired generalization of ρ^α for real α. We consider the Abel equation $\varphi(\rho(t)) = \varphi(t) + 1$ and under some monotonicity assumptions on ρ can find its invertible solution φ. Then using the formula $\rho^\alpha(t) := \varphi^{-1}(\varphi(t) + \alpha)$ we arrive at the introduction of the α-th iteration of ρ for real values α. In particular, if $\rho(t) = q^{-1}(t - h)$, then the corresponding Abel equation admits the functions $\varphi(t) = \log_{q^{-1}}(t + h/(q-1))$ $(q > 1)$ and $\varphi(t) = -t/h$ $(q = 1)$ as the required solutions. Substituting these φ into the formula for ρ^α we obtain the expression

$$\rho^\alpha(t) = q^{-\alpha}(t - [\alpha]_q h), \quad \alpha \in \mathbb{R}, \tag{4.12}$$

which coincides with our expectations concerning the "natural" extension of ρ^α for real α. Quite analogously we can find that $\sigma^\alpha(t) = q^\alpha t + [\alpha]_q h$ for all $\alpha \in \mathbb{R}$.

Considering $t \in \mathbb{T}^{t_0}_{(q,h)}$, it follows immediately from (4.12) that $\rho^\alpha(t) \notin \mathbb{T}^{t_0}_{(q,h)}$ for non-integer values α, which is the second mentioned issue. On this account we introduce the "shifted" time scale

$$\mathbb{T}^{t_0,\alpha}_{(q,h)} := \{t_0 q^{\alpha+k} + [\alpha+k]_q h, k \in \mathbb{Z}\} \cup \left\{\frac{h}{1-q}\right\},$$

where $q \geq 1$, $h \geq 0$, $q + h > 1$ and $\alpha \in \mathbb{R}$. Now we can state the following definition.

Definition 2 (Delta (q,h)-fractional integral). Let $\alpha \in \mathbb{R}^+$, $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$ and let $a \in \mathbb{T}^{t_0}_{(q,h)}$, $t \in \mathbb{T}^{t_0,\alpha}_{(q,h)}$. Then we define the delta (q,h)-fractional integral of f at t by

$$({}_a\Delta^{-\alpha}_{(q,h)} f)(t) := \frac{1}{\Gamma_q(\alpha)} \int_a^{\rho^{\alpha-1}(t)} (t - \sigma^{\alpha-1}(\tau))^{(\alpha-1)}_{(q^{-1},h)} f(\tau)\Delta\tau. \tag{4.13}$$

Similarly as in the nabla case, we can use the following property admitting the alternative introduction of delta (q,h)-fractional integral (the symbol $k(t,\tau)$ has the same meaning as in Lemma 1, but now it is a non-integer provided α is a non-integer):

Lemma 2. *Let $\alpha \in \mathbb{R}^+$, $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$ and let $a \in \mathbb{T}^{t_0}_{(q,h)}$, $t \in \mathbb{T}^{t_0,\alpha}_{(q,h)}$. Then*

$$({}_a\Delta^{-\alpha}_{(q,h)} f)(t) = q^{\binom{\alpha}{2}} \int_a^{\rho^{\alpha-1}(t)} (\mu(\tau))^{\alpha-1} \begin{bmatrix} k(t,\tau) - 1 \\ \alpha - 1 \end{bmatrix}_q f(\tau)\Delta\tau. \tag{4.14}$$

Remark 6. By Lemma 2, $({}_a\Delta^{-\alpha}_{(q,h)} f)(t) = 0$ for all $t \leq \sigma^{\alpha-1}(a)$ and $\alpha \in \mathbb{R}^+ \backslash \mathbb{Z}^+$. If $\alpha \in \mathbb{Z}^+$, then $({}_a\Delta^{-\alpha}_{(q,h)} f)(t) = 0$ for $t = a, \sigma(a), \ldots, \sigma^{\alpha-1}(a)$.

5. Fractional (q,h)-Derivatives

The nabla and delta (q,h)-derivatives of higher integer orders are defined iteratively:

$$\nabla^n_{(q,h)} f := \nabla_{(q,h)} \nabla^{n-1}_{(q,h)} f, \quad \Delta^n_{(q,h)} f := \Delta_{(q,h)} \Delta^{n-1}_{(q,h)} f, \quad n = 2, 3, \ldots.$$

To obtain their more transparent form we can present

Lemma 3. *Let $n \in \mathbb{Z}^+$, $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$ and let $t \in \mathbb{T}^{t_0}_{(q,h)}$, $t > h/(1-q)$. Then*

$$(\nabla^n_{(q,h)} f)(t) = (\nu(t))^{-n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(\rho^k(t)). \tag{5.1}$$

Lemma 4. *Let $n \in \mathbb{Z}^+$, $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$ and let $t \in \mathbb{T}^{t_0}_{(q,h)}$, $t > h/(1-q)$. Then*

$$(\Delta^n_{(q,h)} f)(t) = q^{-\binom{n}{2}} (\mu(t))^{-n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(\sigma^{n-k}(t)). \tag{5.2}$$

Proof. Both formulae can be easily verified by the induction principle. The key step consists in the utilization of q-Pascal rule

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q . \qquad \square$$

Now we can proceed to the introduction of (q,h)-fractional derivatives.

Definition 3 (Nabla (q,h)-fractional derivative). Let $\alpha \in \mathbb{R}^+$, $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$ and let $a, t \in \mathbb{T}^{t_0}_{(q,h)}$. Then we define the nabla (q,h)-fractional derivative of f at t by

$$({}_a\nabla^{+\alpha}_{(q,h)} f)(t) := (\nabla^n_{(q,h)}\, {}_a\nabla^{-(n-\alpha)}_{(q,h)} f)(t), \tag{5.3}$$

where $n \in \mathbb{Z}^+$ is given by $n - 1 < \alpha \leq n$ and we put $({}_a\nabla^{-0}_{(q,h)} f)(t) = f(t)$.

Definition 4 (Delta (q,h)-fractional derivative). Let $\alpha \in \mathbb{R}^+$, $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$ and let $a \in \mathbb{T}^{t_0}_{(q,h)}$, $t \in \mathbb{T}^{t_0,-\alpha}_{(q,h)}$. Then we define delta (q,h)-fractional derivative of f at t by

$$({}_a\Delta^{+\alpha}_{(q,h)} f)(t) := (\Delta^n_{(q,h)}\, {}_a\Delta^{-(n-\alpha)}_{(q,h)} f)(t), \tag{5.4}$$

where $n \in \mathbb{Z}^+$ is given by $n - 1 < \alpha \leq n$ and we put $({}_a\Delta^{-0}_{(q,h)} f)(t) = f(t)$.

Remark 7. Definitions 3 and 4 can be considered as (q,h)-discrete analogues of the Riemann–Liouville fractional derivative (for more details see, e.g. [18]). Note also that fractional derivatives depend on the lower bound a as integrals do (in our definitions, subscript a is used). This dependence vanishes provided derivatives of integer order are considered.

6. Some Basic Properties and Comparisons

This section formulates some basic properties of (q,h)-fractional calculus. In particular, we wish to discuss the validity of the composition rule for (q,h)-fractional integrals and derivatives.

First we present the following expressions of (q,h)-fractional integrals which are more convenient from the computational viewpoint.

Proposition 1. *Let $\alpha \in \mathbb{R}^+$, $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$ and let $a, t \in \mathbb{T}^{t_0}_{(q,h)}$ be such that $t = \sigma^m(a)$ for a suitable $m = m(t) \in \mathbb{Z}^+$. Then*

$$({}_a\nabla^{-\alpha}_{(q,h)} f)(t) = (\nu(t))^\alpha \sum_{k=0}^{m-1} (-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(\rho^k(t)) \,. \tag{6.1}$$

Proposition 2. *Let $\alpha \in \mathbb{R}^+$, $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$ and let $a \in \mathbb{T}^{t_0}_{(q,h)}$, $t \in \mathbb{T}^{t_0,\alpha}_{(q,h)}$ be such that $t = \sigma^{m+\alpha}(a)$ for a suitable $m = m(t) \in \mathbb{Z}^+ \cup \{0\}$. Then*

$$({}_a\Delta^{-\alpha}_{(q,h)} f)(t) = q^{-\binom{-\alpha}{2}} (\mu(t))^\alpha \sum_{k=0}^{m} (-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(\sigma^{-\alpha-k}(t)) \,. \tag{6.2}$$

Proof. Both formulae follow immediately from (4.11) and (4.14) by use of the relations (2.4), (2.3) and the property $\begin{bmatrix} \alpha+k-1 \\ \alpha-1 \end{bmatrix}_q = \begin{bmatrix} \alpha+k-1 \\ k \end{bmatrix}_q = (-1)^k q^{\alpha k + \binom{k}{2}} \begin{bmatrix} -\alpha \\ k \end{bmatrix}_q$. $\square$

The corresponding rewriting of (q,h)-fractional derivatives is a more difficult task.

Proposition 3. *Let $\alpha \in \mathbb{R}^+$, $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$ and let $a, t \in \mathbb{T}^{t_0}_{(q,h)}$ be such that $t = \sigma^m(a)$ for a suitable $m = m(t) \in \mathbb{Z}^+$. Then*

$$({}_a\nabla^{+\alpha}_{(q,h)} f)(t) = \begin{cases} (\nu(t))^{-\alpha} \displaystyle\sum_{k=0}^{m-1} (-1)^k \begin{bmatrix} \alpha \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(\rho^k(t)), & \alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+, \\ (\nu(t))^{-\alpha} \displaystyle\sum_{k=0}^{\alpha} (-1)^k \begin{bmatrix} \alpha \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(\rho^k(t)), & \alpha \in \mathbb{Z}^+. \end{cases} \tag{6.3}$$

Proof. Let $q > 1$, $h \geq 0$ and $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$. It follows from (5.3) and (5.1) that

$$({}_a\nabla^{+\alpha}_{(q,h)} f)(t) = (\nu(t))^{-n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} ({}_a\nabla^{-(n-\alpha)}_{(q,h)} f)(\rho^k(t)). \tag{6.4}$$

If $m < n+1$, then the integrals $({}_a\nabla^{-(n-\alpha)}_{(q,h)} f)(\rho^k(t))$ are zero for $k = m, m+1, \ldots, n$ (see Remark 5). On the other hand, if $m \geq n+1$, then q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is zero for $k > n$. It means that in both cases we may write the upper bound of the sum in (6.4) as $m - 1 = \log_q \nu(t)/\nu(a) - 1$. Then using (6.1) and the relation $\nu(\rho^k(t)) = q^{-k}\nu(t)$ we get

$$\begin{aligned} ({}_a\nabla^{+\alpha}_{(q,h)} f)(t) = (\nu(t))^{-n} &\sum_{k=0}^{\log_q \frac{\nu(t)}{\nu(a)} - 1} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} (\nu(\rho^k(t)))^{n-\alpha} \\ \times &\sum_{j=0}^{\log_q \frac{\nu(\rho^k(t))}{\nu(a)} - 1} (-1)^j \begin{bmatrix} \alpha - n \\ j \end{bmatrix}_q q^{\binom{j}{2}} f(\rho^j(\rho^k(t))) \end{aligned}$$

$$
\begin{aligned}
&= (\nu(t))^{-\alpha} \sum_{k=0}^{\log_q \frac{\nu(t)}{\nu(a)}-1} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} q^{k(\alpha-n)} \\
&\times \sum_{j=0}^{\log_q \frac{\nu(t)}{\nu(a)}-k-1} (-1)^j \begin{bmatrix} \alpha-n \\ j \end{bmatrix}_q q^{\binom{j}{2}} f(\rho^{j+k}(t)) \\
&= (\nu(t))^{-\alpha} \sum_{k=0}^{\log_q \frac{\nu(t)}{\nu(a)}-1} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} q^{k(\alpha-n)} \sum_{j=0}^{\log_q \frac{\nu(t)}{\nu(a)}-k-1} (-1)^{\log_q \frac{\nu(t)}{\nu(a)}-k-j-1} \\
&\times \begin{bmatrix} \alpha-n \\ \log_q \frac{\nu(t)}{\nu(a)}-k-j-1 \end{bmatrix}_q q^{\binom{\log_q \nu(t)/\nu(a)-k-j-1}{2}} f(\sigma^{j+1}(a)) \\
&= (\nu(t))^{-\alpha} \sum_{j=0}^{\log_q \frac{\nu(t)}{\nu(a)}-1} (-1)^{\log_q \frac{\nu(t)}{\nu(a)}-j-1} f(\sigma^{j+1}(a)) \\
&\times \sum_{k=0}^{\log_q \frac{\nu(t)}{\nu(a)}-j-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} \alpha-n \\ \log_q \frac{\nu(t)}{\nu(a)}-k-j-1 \end{bmatrix}_q \\
&\times q^{\binom{k}{2}+\binom{\log_q \nu(t)/\nu(a)-k-j-1}{2}+k(\alpha-n)} \\
&= (\nu(t))^{-\alpha} \sum_{j=0}^{\log_q \frac{\nu(t)}{\nu(a)}-1} (-1)^{\log_q \frac{\nu(t)}{\nu(a)}-j-1} q^{\binom{\log_q \nu(t)/\nu(a)-j-1}{2}} \\
&\times f(\sigma^{j+1}(a)) \sum_{k=0}^{\log_q \frac{\nu(t)}{\nu(a)}-j-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} \alpha-n \\ \log_q \frac{\nu(t)}{\nu(a)}-k-j-1 \end{bmatrix}_q \\
&\times q^{k^2-k\left(\log_q \frac{\nu(t)}{\nu(a)}-j-1\right)+k(\alpha-n)}.
\end{aligned}
$$

The inner sum can be calculated via the identity

$$
\sum_{k=0}^{p} \begin{bmatrix} x \\ p-k \end{bmatrix}_q \begin{bmatrix} y \\ k \end{bmatrix}_q q^{k^2-pk+kx} = \begin{bmatrix} x+y \\ p \end{bmatrix}_q, \quad x, y \in \mathbb{R}, \;\; p \in \mathbb{Z}^+
$$

and on this account we can write $({}_a\nabla^{+\alpha}_{(q,h)} f)(t)$ in the form

$$
\begin{aligned}
&(\nu(t))^{-\alpha} \sum_{j=0}^{\log_q \frac{\nu(t)}{\nu(a)}-1} (-1)^{\log_q \frac{\nu(t)}{\nu(a)}-j-1} q^{\binom{\log_q \nu(t)/\nu(a)-j-1}{2}} f(\sigma^{j+1}(a)) \begin{bmatrix} \alpha \\ \log_q \frac{\nu(t)}{\nu(a)}-j-1 \end{bmatrix}_q \\
&\quad = (\nu(t))^{-\alpha} \sum_{j=0}^{\log_q \frac{\nu(t)}{\nu(a)}-1} (-1)^j \begin{bmatrix} \alpha \\ j \end{bmatrix}_q q^{\binom{j}{2}} f(\rho^j(t)).
\end{aligned}
$$

If $\alpha \in \mathbb{Z}^+$, then we have $({}_a\nabla_{(q,h)}^{-0} f)(t) = f(t)$ and the corresponding result follows immediately.

In the case $q = 1$, $h > 0$ we have $m = (t-a)/h$ and the property $\binom{x+y}{p} = \sum_{k=0}^{p} \binom{x}{p-k}\binom{y}{k}$ implies the required formula quite analogously. $\square$

Proposition 4. *Let $\alpha \in \mathbb{R}^+$, $f : \mathbb{T}_{(q,h)}^{t_0} \to \mathbb{R}$ and let $a \in \mathbb{T}_{(q,h)}^{t_0}$, $t \in \mathbb{T}_{(q,h)}^{t_0,-\alpha}$ be such that $t = \sigma^{m-\alpha}(a)$ for a suitable $m = m(t) \in \mathbb{Z}^+ \cup \{0\}$. Then*

$$({}_a\Delta_{(q,h)}^{+\alpha} f)(t) = \begin{cases} q^{-\binom{\alpha}{2}}(\mu(t))^{-\alpha} \displaystyle\sum_{k=0}^{m} (-1)^k \begin{bmatrix} \alpha \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(\sigma^{\alpha-k}(t)), & \alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+, \\ q^{-\binom{\alpha}{2}}(\mu(t))^{-\alpha} \displaystyle\sum_{k=0}^{\alpha} (-1)^k \begin{bmatrix} \alpha \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(\sigma^{\alpha-k}(t)), & \alpha \in \mathbb{Z}^+. \end{cases} \tag{6.5}$$

Proof. Let $q > 1$, $h \geq 0$ and $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$. By (5.4) and (5.2) we have

$$({}_a\Delta_{(q,h)}^{+\alpha} f)(t) = q^{-\binom{n}{2}}(\mu(t))^{-n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} ({}_a\Delta_{(q,h)}^{-(n-\alpha)} f)(\sigma^{n-k}(t)).$$

If $m < n$ then $n - m$ integrals $({}_a\Delta_{(q,h)}^{-(n-\alpha)} f)(\sigma^{n-k}(t))$ are zero for $k = m+1, m+2, \ldots, n$ with respect to Remark 6. On the other hand, if $m \geq n$, then the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is zero for $k > n$. Hence we conclude that the summation bound n can be replaced by $m = \log_q \mu(t)/\mu(a) + \alpha$ and thus we may write

$$\begin{aligned} ({}_a\Delta_{(q,h)}^{+\alpha} f)(t) &= q^{-\binom{n}{2}}(\mu(t))^{-n} \sum_{k=0}^{\log_q \frac{\mu(t)}{\mu(a)}+\alpha} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} q^{-\binom{-(n-\alpha)}{2}} \\ &\times (\mu(\sigma^{n-k}(t)))^{n-\alpha} \sum_{j=0}^{\log_q \frac{\mu(\sigma^{n-k}(t))}{\mu(a)}-(n-\alpha)} (-1)^j \begin{bmatrix} -(n-\alpha) \\ j \end{bmatrix}_q q^{\binom{j}{2}} f(\sigma^{\alpha-j-k}(t)). \end{aligned}$$

The remaining steps are analogous to those employed in the proof of Proposition 3. $\square$

Remark 8. The assertions of Propositions 1–4 are derived under the assumption that values of a and t are related by a finite iteration of the jump operator. This particularly means that the case, where the lower bound a coincides with the time scale cluster point $h/(1-q)$, is excluded from our considerations. If we admit $a = h/(1-q)$, then the expressions (6.1), (6.2), (6.3) and (6.5) continue to be valid, but the corresponding finite series become the infinite ones as $m \to \infty$ (in this case the convergence of the infinite series is assumed).

Formulae (6.1), (6.3) and (6.2), (6.5) imply that the integrals and derivatives can be written in the unified form (negative values of α stand for integrals while positive ones mean derivatives). The notion "differ-integral" is then commonly used. We also note that for $q = 1$ the results (6.1), (6.3) are closely related to Grünwald–Letnikov approach to fractional differentiating in the continuous case (see e.g. [15] and [18]).

Now we present some basic properties of (q,h)-fractional integrals and derivatives. The following linearity property is an immediate consequence of their definitions.

Lemma 5. *Let $\alpha \in \mathbb{R}^+$, $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$, $g : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$ and let c_1, $c_2 \in \mathbb{R}$. Then*

$$({}_a\nabla^{\pm\alpha}_{(q,h)}(c_1 f + c_2 g))(t) = c_1\,({}_a\nabla^{\pm\alpha}_{(q,h)} f)(t) + c_2\,({}_a\nabla^{\pm\alpha}_{(q,h)} g)(t), \quad a, t \in \mathbb{T}^{t_0}_{(q,h)},$$

$$({}_a\Delta^{\pm\alpha}_{(q,h)}(c_1 f + c_2 g))(t) = c_1({}_a\Delta^{\pm\alpha}_{(q,h)} f)(t) + c_2({}_a\Delta^{\pm\alpha}_{(q,h)} g)(t), \quad a \in \mathbb{T}^{t_0}_{(q,h)}, \quad t \in \mathbb{T}^{t_0,\pm\alpha}_{(q,h)}.$$

The next assertions discuss the validity of the composition rule. Their proofs utilize the similar line of arguments as given in the proof of Proposition 3 and therefore are omitted.

Theorem 3. *Let α, $\beta \in \mathbb{R}^+$ and $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$. Then*

$$({}_a\nabla^{-\alpha}_{(q,h)}\,{}_a\nabla^{-\beta}_{(q,h)} f)(t) = ({}_a\nabla^{-(\alpha+\beta)}_{(q,h)} f)(t)$$

provided a, $t \in \mathbb{T}^{t_0}_{(q,h)}$ and

$$({}_{\sigma^\beta(a)}\Delta^{-\alpha}_{(q,h)}\,{}_a\Delta^{-\beta}_{(q,h)} f)(t) = ({}_a\Delta^{-(\alpha+\beta)}_{(q,h)} f)(t)$$

provided $a \in \mathbb{T}^{t_0}_{(q,h)}$, $t \in \mathbb{T}^{t_0,\alpha+\beta}_{(q,h)}$.

Theorem 4. *Let $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}^+\backslash\mathbb{Z}^+$ and $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$. Then*

$$({}_a\nabla^{+\alpha}_{(q,h)}\,{}_a\nabla^{+\beta}_{(q,h)} f)(t) = ({}_a\nabla^{+(\alpha+\beta)}_{(q,h)} f)(t)$$

provided a, $t \in \mathbb{T}^{t_0}_{(q,h)}$ and

$$({}_{\rho^\beta(a)}\Delta^{+\alpha}_{(q,h)}\,{}_a\Delta^{+\beta}_{(q,h)} f)(t) = ({}_a\Delta^{+(\alpha+\beta)}_{(q,h)} f)(t)$$

provided $a \in \mathbb{T}^{t_0}_{(q,h)}$, $t \in \mathbb{T}^{t_0,-\alpha-\beta}_{(q,h)}$.

Remark 9. The composition rule does not hold if the inner derivative is of an integer order. It follows from the fact that such a derivative is no longer the function of the upper integration bound. To illustrate this by a counterexample, one can easily verify that considering $\beta = 1$ and any $\alpha \in \mathbb{R}^+\backslash\mathbb{Z}^+$, the application of the nabla formula (6.3) yields

$$({}_a\nabla^{+\alpha}_{(q,h)}\nabla_{(q,h)} f)(t) = ({}_a\nabla^{+(\alpha+1)}_{(q,h)} f)(t) - (-1)^{m-1}\begin{bmatrix}\alpha\\ m-1\end{bmatrix}_q f(a), \quad t = \sigma^m(a).$$

Considering $t > a = h/(1-q)$ the composition rule for derivatives holds without any restrictions (i.e. also in the case where the inner derivative is of an integer order).

7. Concluding Remarks

In the final section we mention some comparisons demonstrating that our definitions and results on (q,h)-fractional integrals and derivatives are consistent with the standard introductions and properties of these notions in the quantum and difference calculus. As we declared earlier, one of our goals was the unification of various definitions and approaches appearing in the above cited papers (and also in other references). We illustrate by several examples that our definitions actually unify these approaches.

The fractional q-integral is usually defined for $0 < q < 1$ via the formula

$$(\nabla_q^{-\alpha} f)(t) := \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - q\tau)_q^{(\alpha-1)} f(\tau) \nabla\tau, \tag{7.1}$$

(see, e.g. [4]). It is easy to check that (7.1) follows from (4.10) by the choice $a = h = 0$ and $q > 1$. Further, considering the time scale $\mathbb{T}^1_{(q,0)}$, $q > 1$ and employing the change of variables $t = q^n$, $f_n = f(t) = f(q^n)$ in (6.1) (we recall that this relation is equivalent to (4.10)), we obtain the nabla fractional integral

$$_1\nabla^{-\alpha}_{(q,0)} f_n = q^{(n-1)\alpha}(q-1)^{\alpha} \sum_{k=0}^{n-1} (-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} f_{n-k},$$

which is the form presented and investigated in [16]. Similarly, the nabla fractional differences (the case $q = h = 1$) have been studied and introduced in [10, 11], where, among others, the conditions for the validity of the exponential law have been derived. Then basic properties of nabla (q,h)-fractional integrals and derivatives stated in Sec. 6 generalize and extend some parts of the corresponding papers.

The definitions of delta fractional integrals and derivatives are usually employed rarely. Probably the main disadvantage of these delta definitions consists in the fact that they require points t not belonging to the given time scale. Introduction and investigation of delta fractional sums and differences is the subject of papers [3], [7] and [14], where some interesting properties have been reported. Setting $q = h = 1$ in (4.13) and (5.4), we can again observe the consistency with the relevant definitions presented in these papers (e.g. (4.13) becomes just the defining relation (2.1) of [3]). Furthermore, our discussions on the validity of the composition rule for fractional derivatives extend the known results on this topic in the delta (as well as nabla) case. Of course, the extension of other results derived in the relevant papers is a matter of further investigations.

The second purpose of this contribution was to outline a possible extension of the fractional calculus basics to more general time scales. We performed this extension to the (q,h)-fractional calculus, but the extension to other time scales (especially those with a nonlinear graininess) remains open. We note that some of the tools utilized in our investigations are applicable to arbitrary time scales. It concerns especially particular steps (e.g. the relation (4.4) is of general validity as well as the procedure for the determination of non-integer iterations of jump operators), but partially also the entire assertions (e.g. Lemmas 3 and 4 can be, after some nontrivial calculations, extended to any time scale). On the other hand, the extension of the key mathematical tools in the fractional calculus, especially the Cauchy formula, to arbitrary time scale does not seem to be obvious. This and other related questions are assumed to be the subject of the next research.

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