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LAPLACE INVARIANTS FOR GENERAL HYPERBOLIC SYSTEMS

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We consider the generalization of Laplace invariants to linear differential systems of arbitrary rank and dimension. We discuss completeness of certain subsets of invariants.

Keywords: Laplace invariants; covariants; differential invariants; hyperbolic systems.

1. Introduction

The classical Laplace invariants [10] were introduced in the context of second-order, linear hyperbolic systems of the form

$$z_{,xy} + az_{,x} + bz_{,y} + cz = 0, \tag{1.1}$$

where a, b and c are given functions and $z = z(x, y)$ is an unspecified solution of this partial differential equation.

The *form* of Eq. (1.1) is unchanged under a general transformation $z \mapsto g(x, y)z$ where $g(x, y)$ is a sufficiently differentiable, but otherwise arbitrary, function. In fact the coefficients of the equation are simply mapped into new functions,

$$\begin{aligned} a &\mapsto a' = a + g^{-1}g_{,y}\,, \\ b &\mapsto b' = b + g^{-1}g_{,x}\,, \\ c &\mapsto c' = c + g^{-1}ag_{,x} + g^{-1}bg_{,y} + g^{-1}g_{,xy} \end{aligned} \tag{1.2}$$

and it is easily seen that the following two functions are *invariant* under such a transformation:

$$h = a_{,x} + ab - c, \tag{1.3}$$

$$k = b_{,y} + ab - c. \tag{1.4}$$

More than this, the pair $\{h, k\}$ is a *complete* set of invariants in that two equations of the form (1.1) having exactly the same invariants, as functions of x and y, must necessarily be related by a gauge transformation of the sort described. The family of equations is thus partitioned into equivalence classes labeled by these pairs of functions. These functions are called Laplace invariants by many researchers in integrability theory (see e.g. [5, 6, 8, 14, 16, 17, 19]).

Such invariants have played an important role in recent work on the geometrical theory of integrable systems and soliton equations. It is not our purpose to rehearse these connections here and we refer the interested reader to Refs. [11, 20] where much of the material is reviewed. However, it is important to point out that a valuable role is played by the *Laplace map*, a differential map between equations of the form (1.1) which acts on the equivalence classes according to the equations of the two-dimensional Toda lattice [12, 18]. The generalization of the Laplace map to higher dimension and higher rank linear systems is of great importance [1, 20]. This paper should be regarded as a prolegomenon to a general theory of such transformations.

There are generalizations of the classical Laplace invariants already in existence. They are mainly developed to answer questions about factorizability and integrability of partial differential equations.

Thus in the work of [8, 14, 17] Laplace invariants are constructed for scalar linear partial differential operators of order greater than two. The accompanying calculations are computationally demanding and attempts have been made to put the construction into a more abstract, systematic framework [10, 13].

Another important direction of generalization is to nonlinear systems. In [4] are defined invariants for Toda lattices via a hyperbolic linearization and in [7] a geometric view is taken via exterior differential algebra. Here the invariants are defined via localization and this may be related to the approach of the previous paragraph.

Laplace invariants arise naturally in the theory of the Toda Lattice and in integrable equations of hydrodynamic type [3, 15].

There is a large body of work on the definition and application of such invariants and there is no space to be comprehensive here. Further work is cited in the references given.

In Sec. 2 we develop a complete theory of generalized Laplace invariants for n-dimensional, linear hyperbolic systems. In particular we describe a minimal, complete set of invariants labeling gauge-equivalence classes. The implementation of Laplace maps on such invariants will form the subject of another study.

In Sec. 3 we consider the case of an n-dimensional matrix system with fewer than n independent variables. In this case the scalar coefficients of the first section are effectively replaced by matrix-valued objects. We are able to give a general prescription for constructing matrix-valued, generalized Laplace covariants generalizing that of, say, [9]. Invariants in such circumstances would arise from traces of suitable polynomials in the covariants. We do not discuss completeness in this case and again we do not here treat the corresponding generalized Laplace maps.

Before proceeding let us note that the form (1.1), though symmetric, has a degree of redundancy about it. We may choose to transform it using a gauge transformation $z \mapsto gz$ where g satisfies $g_{,y} = -a(x,y)g$. In this case the transformed equation is

$$z_{,xy} + \left(\int \{k - h\} dy\right) z_{,y} - hz = 0, \tag{1.5}$$

and the dependence on the equivalence class is explicit. An equation of this reduced form,

$$z_{,xy} + bz_{,y} + cz = 0, \tag{1.6}$$

still retains a gauge covariance, namely $z \mapsto g(x)z$, the gauge function depending upon x alone and it is naturally written as a system in z and $z_{,y}$:

$$\begin{pmatrix} \partial_x & -\beta c \\ 1/\beta & \partial_y \end{pmatrix} \begin{pmatrix} -\beta z_{,y} \\ z \end{pmatrix} = 0, \tag{1.7}$$

where $\beta_{,x} = \beta b$.

Of course, we might equally consider reduced forms

$$z_{,xy} + az_{,x} + cz = 0, \tag{1.8}$$

with y dependent gauge transformations, but what we cannot do in general is reduce to the form

$$z_{,xy} + cz = 0, \tag{1.9}$$

as this requires that the special relationship $h = k$, $\forall x, y$ should hold.

Equally we could *start* with a general system form

$$\begin{pmatrix} \partial_x + h_{11} & h_{12} \\ h_{21} & \partial_y + h_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0 \tag{1.10}$$

as is done in [1]. Gauge transformations preserving this form of system are 2×2 diagonal matrices acting on the two component vector of the z_i. The gauge invariants are

$$(12) = h_{12}h_{21}, \tag{1.11}$$

$$[12] = h_{11,y} - h_{22,x} + \frac{1}{2} \ln \left(\frac{h_{12}}{h_{21}}\right)_{,xy}. \tag{1.12}$$

However the redundancy is also present here and we can use the gauge transformation to kill the diagonal terms h_{11} and h_{22}. This leaves us with the canonical form

$$\begin{pmatrix} \partial_x & h_{12} \\ h_{21} & \partial_y \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0, \tag{1.13}$$

and residual gauge transformations

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} g_1(y) & 0 \\ 0 & g_2(x) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{1.14}$$

with invariants

$$(12) = h_{12}h_{21}, \tag{1.15}$$

$$[12] = \frac{1}{2}\ln\left(\frac{h_{12}}{h_{21}}\right)_{,xy}. \tag{1.16}$$

It is not difficult to verify that these invariants are a *complete* set for the canonical form (1.13).

In what follows we shall consider $n \times n$ systems and discuss the completeness of the sets of invariants constructed in a similar manner to those presented in this introduction. We shall also relate them to second-order, *matrix* equations, i.e. those of the type (1.1) but having a, b and c as square matrices rather than simple functions.

We use the word *dimension* to denote the number of independent variables which we shall henceforth write as $x_1, x_2, \ldots, x_n$. By *rank* we shall understand the number of components in the solution vector z: $z_1, z_2, \ldots, z_r$.

2. Invariants for General Hyperbolic Systems

Definition 2.1. Let $\mathbb{L}$ be an $n \times n$ matrix differential operator

$$\mathbb{L} = \begin{pmatrix} \partial_1 + h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & \partial_2 + h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \cdots & \partial_n + h_{nn} \end{pmatrix},$$

where ∂_i stands for $\partial/\partial x_i$ and the h_{ij} are functions of $x_1, x_2, \ldots, x_n$. If g is a diagonal $n \times n$ matrix such that g^{-1} exists, then $H = H(h_{ij})$ is *invariant* under the gauge transformation

$$\mathbb{L}' = g^{-1}\mathbb{L}g,$$

so long as $H(h'_{ij}) = H(h_{ij})$.

2.1. *The case where rank and dimension are equal*

In this case we deal with matrix differential operators

$$\mathbb{L} = \begin{pmatrix} \partial_1 & 0 & \cdots & 0 \\ 0 & \partial_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & \partial_n \end{pmatrix} + \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & & h_{2n} \\ \vdots & & \ddots & \vdots \\ h_{n1} & \cdots & & h_{nn} \end{pmatrix} \tag{2.1}$$

and gauge transformations

$$\mathbb{L} \mapsto \mathbb{L}' = g^{-1}\mathbb{L}g \tag{2.2}$$

of the form

$$g = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & g_n \end{pmatrix}. \tag{2.3}$$

The h_{ij} and g_i here are functions of all variables $x_1, x_2, \ldots, x_n$ but we may choose the reduced (canonical) form in which the diagonal entries $h_{11}, h_{22}, \ldots, h_{nn}$ are gauged away by solving the n equations: $g_{i,i} + h_{ii}g_i = 0$, $i = 1, 2, \ldots, n$.

$$\mathbb{L} = \begin{pmatrix} \partial_1 & 0 & \cdots & 0 \\ 0 & \partial_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & \partial_n \end{pmatrix} + \begin{pmatrix} 0 & h_{12} & \cdots & h_{1n} \\ h_{21} & 0 & & h_{2n} \\ \vdots & & \ddots & \vdots \\ h_{n1} & \cdots & & 0 \end{pmatrix}. \tag{2.4}$$

The residual gauge freedom is

$$g = \begin{pmatrix} g_1(\hat{x}_1) & 0 & \cdots & 0 \\ 0 & g_2(\hat{x}_2) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & g_n(\hat{x}_n) \end{pmatrix} \tag{2.5}$$

where hatted variables are deleted from the list of arguments in each g_i. Under such transformations

$$h_{ij} \mapsto g_i(\hat{x}_i)^{-1} g_j(\hat{x}_j) h_{ij} \tag{2.6}$$

and it is easily seen that the following objects are all invariant: Choose from the n labels $\{1, 2, \ldots, n\}$ a subset of p distinct ones, $\{i_1, i_2, \ldots, i_p\}$, and define the symbol:

$$(i_1 i_2 \cdots i_p) = h_{i_1 i_2} h_{i_2 i_3} \cdots h_{i_p i_1}. \tag{2.7}$$

We say the symbol $(i_1 i_2 \cdots i_p)$ has *length* p. Thus in the case of the symbols of lengths 2 and 3 we have $(ij) = h_{ij}h_{ji}$ and $(ijk) = h_{ij}h_{jk}h_{ki}$.

Because of the cyclic symmetry in these products there will be $\frac{n!}{p(n-p)!}$ symbols of length p. The symbols of length p are permuted under the action of S_n, the symmetric group on n labels.

In addition there are $\frac{1}{2}n(n-1)$ invariants denoted by square bracket symbols thus:

$$[ij] = -[ji] = \frac{1}{2}\partial_i \partial_j \ln\left(\frac{h_{ij}}{h_{ji}}\right). \tag{2.8}$$

We call the invariants (2.7) and (2.8) *simple*. All functions of these symbols are themselves invariant but we will now show that within the set of *simple* invariants there are a *complete* subset i.e. a set the knowledge of which is enough to determine the operator $\mathbb{L}$ completely up to gauge transformations.

Lemma 2.2. *The functions* $[ij]$ *and* $(i_1 i_2 \cdots i_p)$ *are invariants.*

Proof. We consider the $n \times n$ differential operator matrix $\mathbb{L}$

$$\mathbb{L} = \begin{pmatrix} \partial_1 & h_{12} & \cdots & h_{1n} \\ h_{21} & \partial_2 & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \cdots & \partial_n \end{pmatrix},$$

where h_{ij} are functions of $x_1, x_2, \ldots, x_n$. We find the invariants of $\mathbb{L}$ by using the gauge transformation, $g^{-1}\mathbb{L}g = \mathbb{L}'$, where g is a $n \times n$ diagonal matrix

$$g = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_n \end{pmatrix}.$$

Then $g^{-1}\mathbb{L}g = \mathbb{L}'$ gives us

$$0 = (\ln g_i)_{,i}, \tag{2.9}$$

$$h'_{ij} = g_i^{-1} g_j h_{ij}, \quad (i \neq j). \tag{2.10}$$

Now

$$\frac{1}{2}\left(\ln \frac{h'_{ij}}{h'_{ji}}\right)_{,ij} = \frac{1}{2}\left(\ln \frac{h_{ij}}{h_{ji}}\right)_{,ij} + \left(\ln \frac{g_j}{g_i}\right)_{,ij}$$

which gives

$$\frac{1}{2}\left(\ln \frac{h'_{ij}}{h'_{ji}}\right)_{,ij} = \frac{1}{2}\left(\ln \frac{h_{ij}}{h_{ji}}\right)_{,ij}$$

since

$$g_{r,r} = 0, \tag{2.11}$$

where $r = i, j$. This gives us the antisymmetric invariants

$$[ij] = \frac{1}{2}\left(\ln \frac{h_{ij}}{h_{ji}}\right)_{,ij}. \tag{2.12}$$

Finally we consider the following relations

$$\begin{aligned} h'_{i_1 i_2} &= g_{i_1}^{-1} g_{i_2} h_{i_1 i_2}, \\ h'_{i_2 i_3} &= g_{i_2}^{-1} g_{i_3} h_{i_2 i_3}, \\ h'_{i_3 i_4} &= g_{i_3}^{-1} g_{i_4} h_{i_3 i_4}, \\ &\vdots \\ h'_{i_{p-1} i_p} &= g_{i_{p-1}}^{-1} g_{i_p} h_{i_{p-1} i_p}, \\ h'_{i_p i_1} &= g_{i_p}^{-1} g_{i_1} h_{i_p i_1}. \end{aligned}$$

Then we obtain

$$h'_{i_1 i_2} h'_{i_2 i_3} h'_{i_3 i_4} \cdots h'_{i_{p-1} i_p} h'_{i_p i_1} = h_{i_1 i_2} h_{i_2 i_3} h_{i_3 i_4} \cdots h_{i_{p-1} i_p} h_{i_p i_1}$$

to give the p-index invariants:

$$(i_1 i_2 i_3 \cdots i_p) = h_{i_1 i_2} h_{i_2 i_3} h_{i_3 i_4} \cdots h_{i_{p-1} i_p} h_{i_p i_1}, \tag{2.13}$$

where the i_r are a choice of p distinct integers in $\{1, 2, \ldots, n\}$.

By recalling (2.12) and (2.13) we now collect all the invariants of $\mathbb{L}$ as follows:

$$[ij] = \frac{1}{2}\left(\ln \frac{h_{ij}}{h_{ji}}\right)_{,ij},$$
$$(i_1 i_2 i_3 \cdots i_p) = h_{i_1 i_2} h_{i_2 i_3} h_{i_3 i_4} \cdots h_{i_{p-1} i_p} h_{i_p i_1}. \qquad \square$$

Definition 2.3. The functions $[ij]$ and $(i_1 i_2 i_3 \cdots i_p)$ are called the *simple* invariants of $\mathbb{L}$.

Theorem 2.4. *The simple invariants form a complete set for the equivalence class of* $\mathbb{L}$ *under gauge transformations, where* $\mathbb{L}$ *is defined by* (2.4).

Proof. The proof depends on showing that one can construct a suitable gauge matrix g. In other words we need to show that

$$\mathbb{L}' = g^{-1}\mathbb{L}g \Leftrightarrow \left\{ \begin{matrix} [ij]' = [ij] \\ (i_1 i_2 i_3 \cdots i_p)' = (i_1 i_2 i_3 \cdots i_p) \end{matrix} \right\},$$

where $\{i_1, i_2, i_3, \ldots, i_p\} \subset \{1, 2, \ldots, n\}$.

We already know that "$\Rightarrow$" is true. We only need to prove the "$\Leftarrow$" part. Assume the RHS is true i.e.

$$[ij]' = [ij],$$
$$(i_1 i_2 i_3 \cdots i_p)' = (i_1 i_2 i_3 \cdots i_p),$$

for all subsets $\{i_1, i_2, \ldots, i_p\} \subseteq \{1, 2, \ldots, n\}$. Let us choose an $n \times n$ diagonal matrix f such that

$$f = \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_n \end{pmatrix},$$

where

$$f_1 = h_{12} h_{23} h_{34} \cdots h_{n-1n},$$
$$f_2 = h'_{12} h_{23} h_{34} \cdots h_{n-1n},$$
$$f_3 = h'_{12} h'_{23} h_{34} \cdots h_{n-1n},$$
$$\vdots$$
$$f_{n-1} = h'_{12} h'_{23} h'_{34} \ldots h'_{n-2n-1} h_{n-1n},$$
$$f_n = h'_{12} h'_{23} h'_{34} \ldots h'_{n-2n-1} h'_{n-1n}.$$

Then we obtain

$$f^{-1}\mathbb{L}f = \begin{pmatrix} \partial_1 + \tilde{h}_{11} & \tilde{h}_{12} & \cdots & \tilde{h}_{1n} \\ \tilde{h}_{21} & \partial_2 + \tilde{h}_{22} & \cdots & \tilde{h}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{h}_{n1} & \tilde{h}_{n2} & \cdots & \partial_n + \tilde{h}_{nn} \end{pmatrix},$$

where

$$\tilde{h}_{ij} = f_i^{-1} f_j h_{ij} \quad (i \neq j),$$
$$\tilde{h}_{ii} = (\ln f_i)_{,i} \quad (i = 1, 2, \ldots, n).$$

Thus we need to show

$$\tilde{h}_{ij} = h'_{ij} \quad (i \neq j). \tag{2.14}$$

We easily prove (2.14) as follows:

$$\tilde{h}_{ij} = f_i^{-1} f_j h_{ij} \quad (i \neq j).$$

Let $i < j$. Then

$$f_i = h'_{12} h'_{23} \cdots h'_{i-1i} h_{ii+1} \cdots h_{j-1j} h_{jj+1} \cdots h_{n-1n},$$
$$f_j = h'_{12} h'_{23} \cdots h'_{i-1i} h'_{ii+1} \cdots h'_{j-1j} h_{jj+1} \cdots h_{n-1n}.$$

Thus

$$\begin{aligned}
\tilde{h}_{ij} = \frac{f_j}{f_i} h_{ij} &= \frac{h'_{ii+1} h'_{i+1i+2} \cdots h'_{j-1j}}{h_{ii+1} h_{i+1i+2} \cdots h_{j-1j}} h_{ij} \\
&= \frac{h'_{ii+1} h'_{i+1i+2} \cdots h'_{j-1j} h'_{ji}}{h_{ii+1} h_{i+1i+2} \cdots h_{j-1j} h_{ji}} \cdot \frac{h_{ji}}{h'_{ji}} \cdot h_{ij} \\
&= \frac{(ii+1i+2 \cdots j)'}{(ii+1i+2 \cdots j)} \cdot \frac{(ij)}{h'_{ji}} \cdot \frac{h'_{ij}}{h'_{ij}} \\
&= \frac{(ij)}{(ij)'} \cdot h'_{ij} \\
&= h'_{ij}
\end{aligned}$$

since $(ii+1i+2 \cdots j)' = (ii+1i+2 \cdots j)$ and $(ij)' = (ij)$.

Similarly

$$\begin{aligned}
\tilde{h}_{ji} = \frac{f_i}{f_j} h_{ji} &= \frac{h_{ii+1} h_{i+1i+2} \cdots h_{j-1j}}{h'_{ii+1} h'_{i+1i+2} \cdots h'_{j-1j}} h_{ji} \\
&= \frac{(ii+1i+2 \cdots j)}{(ii+1i+2 \cdots j)'} h'_{ji} \\
&= h'_{ji}.
\end{aligned}$$

Hence for $i \neq j$ we obtain

$$\tilde{h}_{ij} = h'_{ij}.$$

So we have

$$f^{-1}\mathbb{L}f = \begin{pmatrix} \partial_1 + \tilde{h}_{11} & h'_{12} & \cdots & h'_{1n} \\ h'_{21} & \partial_2 + \tilde{h}_{22} & \cdots & h'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h'_{n1} & h'_{n2} & \cdots & \partial_n + \tilde{h}_{nn} \end{pmatrix},$$

where

$$\tilde{h}_{ii} = (\ln f_i)_{,i}. \tag{2.15}$$

We now need to seek a single function θ so that

$$\theta^{-1}(f^{-1}\mathbb{L}f)\theta = \mathbb{L}'.$$

This requires that θ satisfy the following equations:

$$\theta^{-1}\theta_{,i} + \tilde{h}_{ii} = 0$$

i.e.

$$\theta_{,i} = -\tilde{h}_{ii}\theta.$$

The above equations are consistent $\Leftrightarrow (\theta_{,i})_{,j} = (\theta_{,j})_{,i}$, which gives

$$\tilde{h}_{ii,j} = \tilde{h}_{jj,i}. \tag{2.16}$$

Recalling (2.15) we write

$$\tilde{h}_{ii} = (\ln f_i)_{,i},$$
$$\tilde{h}_{jj} = (\ln f_j)_{,j},$$

and if we substitute these into Eq. (2.16) we obtain

$$[ij]' = [ij]$$

since

$$\left(\frac{f_i}{f_j}\right)^2 = \frac{h_{ij}}{h_{ji}}\frac{h'_{ji}}{h'_{ij}},$$

where

$$\frac{f_i}{f_j} = \frac{h_{ij}}{h'_{ij}} = \frac{h'_{ji}}{h_{ji}}.$$

So the equality of invariants guarantees that the Frobenius integrability condition is satisfied: there exists a function θ such that $\theta^{-1}(f^{-1}\mathbb{L}f)\theta = \mathbb{L}'$, i.e.

$$g^{-1}\mathbb{L}g = \mathbb{L}',$$

where $g = \theta f$. Hence the given invariants of $\mathbb{L}$ are a complete set. □

It should be noted that the simple invariants are not algebraically independent. For instance,

$$(ijk)(ikj) = (ij)(jk)(ki) \tag{2.17}$$

so that there must be a smallest set of simple invariants which is still complete. A *minimal complete* set is given in the following result.

Theorem 2.5. *The simple invariants* $(1i)$, $[ij]$ *and* $(1ij)$ *form a minimal complete set.*

First we prove some lemmas.

Lemma 2.6. *We consider a simple invariant of length* m

$$(i_1 i_2 i_3 \cdots i_{m-1} i_m) = h_{i_1 i_2} h_{i_2 i_3} h_{i_3 i_4} \cdots h_{i_{m-1} i_m} h_{i_m i_1}. \tag{2.18}$$

Let m *be a positive integer such that* $m \geq 4$. *Then*

$$(i_1 i_2 i_3 \cdots i_{m-1} i_m) = \frac{(i_1 i_2 i_3 \cdots i_{m-1})(i_1 i_{m-1} i_m)}{(i_1 i_{m-1})}. \tag{2.19}$$

Proof.

$$\begin{aligned}
RHS &= \frac{(i_1 i_2 i_3 \cdots i_{m-1})(i_1 i_{m-1} i_m)}{(i_1 i_{m-1})} \\
&= \frac{h_{i_1 i_2} h_{i_2 i_3} \cdots h_{i_{m-2} i_{m-1}} h_{i_{m-1} i_1} \cdot h_{i_1 i_{m-1}} h_{i_{m-1} i_m} h_{i_m i_1}}{h_{i_1 i_{m-1}} h_{i_{m-1} i_1}} \\
&= h_{i_1 i_2} h_{i_2 i_3} \cdots h_{i_{m-2} i_{m-1}} h_{i_{m-1} i_m} h_{i_m i_1} \\
&= (i_1 i_2 i_3 \cdots i_{m-1} i_m) = LHS.
\end{aligned}$$

Hence we can replace simple invariants of length $m \geq 4$ with invariants of length $m-1$ up to multiples of invariants of lengths 2 and 3. □

Lemma 2.7. *Let* i, j, k *be three positive integers such that* $i \neq j \neq k$. *Then*

$$(ij) = \frac{(1ij)(1ji)}{(1i)(1j)}, \tag{2.20}$$

$$(ijk) = \frac{(1ij)(1jk)(1ki)}{(1i)(1j)(1k)}. \tag{2.21}$$

Proof.

$$\begin{aligned}
\frac{(1ij)(1ji)}{(1i)(1j)} &= \frac{h_{1i} h_{ij} h_{j1} \cdot h_{1j} h_{ji} h_{i1}}{h_{1i} h_{i1} \cdot h_{1j} h_{j1}} \\
&= h_{ij} h_{ji} = (ij) = LHS.
\end{aligned}$$

Similarly

$$\frac{(1ij)(1jk)(1ki)}{(1i)(1j)(1k)} = \frac{h_{1i}h_{ij}h_{j1} \cdot h_{1j}h_{jk}h_{k1} \cdot h_{1k}h_{ki}h_{i1}}{h_{1i}h_{i1} \cdot h_{1j}h_{j1} \cdot h_{1k}h_{k1}}$$
$$= h_{ij}j_{jk}h_{ki}$$
$$= (ijk) = LHS. \qquad \square$$

Lemma 2.8. *The invariants* $(1ij)$ *are irreducible* (*i.e. they cannot be written purely in terms of invariants with length* 2).

Proof. We will prove this by contradiction. So assume $(1ij)$ is reducible. Thus $(1ij)$ can be expressed in terms of the invariants $(1i)$, $(1j)$ and (ij). So let

$$(1ij) = F[(1i), (1j), (ij)]. \tag{2.22}$$

If we differentiate Eq. (2.22) with respect to h_{i1}, h_{1j} and h_{ji} respectively we obtain the following partial differential equations:

$$0 = \frac{\partial(1ij)}{\partial h_{i1}} = \frac{\partial F}{\partial(1i)} \cdot h_{1i},$$
$$0 = \frac{\partial(1ij)}{\partial h_{1j}} = \frac{\partial F}{\partial(1j)} \cdot h_{j1},$$
$$0 = \frac{\partial(1ij)}{\partial h_{ji}} = \frac{\partial F}{\partial(ij)} \cdot h_{ij}$$

since $(1ij) = h_{1i}h_{ij}h_{j1}$ is independent of h_{i1}, h_{1j} and h_{ji}.

Thus, we find

$$\frac{\partial F}{\partial(1i)} = 0,$$
$$\frac{\partial F}{\partial(1j)} = 0,$$
$$\frac{\partial F}{\partial(ij)} = 0$$

since $h_{1i} \neq 0, h_{j1} \neq 0$ and $h_{ij} \neq 0$.

This shows that $(1ij) =$ constant. This is a contradiction. Therefore the invariant $(1ij)$ is irreducible. □

Proof of Theorem 2.5. We have considered the following simple invariants of length m:

$$(i_1i_2i_3 \cdots i_{m-1}i_m) = h_{i_1i_2}h_{i_2i_3}h_{i_3i_4} \cdots h_{i_{m-1}i_m}h_{i_mi_1}.$$

First we have shown (Lemma 2.6) that these invariants can be reduced up to length 3 and then we have shown (Lemma 2.7) that the invariant (ij) can be written in terms of the simple invariants $(1i)$ and $(1ij)$ and we have also proved that the simple invariant (ijk) can be expressed in terms of the invariants $(1i)$ and $(1ij)$. Finally, we have proved (Lemma 2.8) that the invariant $(1ij)$ is not reducible, in other words, it cannot be reduced to the invariant of length 2.

Hence the proof of the theorem is complete and the result follows: Any invariant of length m can be written in terms of the minimal invariants $(1i)$ and $(1ij)$ where these minimal invariants together with $[ij]$ form a complete set. □

3. Matrix Covariants for General Hyperbolic Systems

3.1. *Matrix covariants*

Let us consider the system

$$Lz = (\partial_x\partial_y + a\partial_x + b\partial_y + c)z = 0, \tag{3.1}$$

where a, b and c are $m \times m$ square matrices. This case is considered in [9]. The gauge transformation on the differential operator L is $L' = g^{-1}Lg$, where g is a $m \times m$ diagonal matrix which gives

$$\begin{aligned} h &= a_{,x} + ba - c, \\ k &= b_{,y} + ab - c, \end{aligned} \tag{3.2}$$

where h and k are gauge *covariants* for the system (3.1): $h' = g^{-1}hg$, $k' = g^{-1}kg$. These covariants are sometimes called invariants in the literature [9].

3.2. *Matrix covariants for* $\mathbb{L}$

Let us consider $\mathbb{L}$ as a $(m_1 + m_2) \times (m_2 + m_1)$ differential matrix operator such that

$$\mathbb{L} = \begin{pmatrix} \partial_1 + h_{11} & h_{12} \\ h_{21} & \partial_2 + h_{22} \end{pmatrix},$$

where $h_{11} \in M_{m_1m_1}$, $h_{12} \in M_{m_1m_2}$, $h_{21} \in M_{m_2m_1}$, $h_{22} \in M_{m_2m_2}$ and $M_{m_im_j}$ is the set of $m_i \times m_j$ matrices.

Strictly speaking we should write $I_{m_1}\partial_1$ and $I_{m_2}\partial_2$ for the differential operator entries where I_{m_1}, I_{m_2} are the unit matrices of dimensions m_1 and m_2. This should be understood in what follows.

The "gauge" transformation g on $\mathbb{L}$ is $\mathbb{L}' = g^{-1}\mathbb{L}g$ for

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},$$

where $g_1 \in M_{m_1m_1}$ and $g_2 \in M_{m_2m_2}$ are both invertible square matrix functions of x_1, x_2. Under this action, $\mathbb{L}' = g^{-1}\mathbb{L}g$, we have

$$\begin{aligned} h'_{11} &= g_1^{-1}h_{11}g_1 + g_1^{-1}g_{1,1}, \\ h'_{12} &= g_1^{-1}h_{12}g_2, \\ h'_{21} &= g_2^{-1}h_{21}g_1, \\ h'_{22} &= g_2^{-1}h_{22}g_2 + g_2^{-1}g_{2,2}. \end{aligned}$$

3.3. *Definitions*

We call an object H of *type* $G_i \times G_j$ if $H' = g_i^{-1} H g_j$. Therefore h_{12} is of type $G_1 \times G_2$ and h_{21} is of type $G_2 \times G_1$. Covariants are of type $G_i \times G_i$. In other words H is a *covariant* if $H' = g_i^{-1} H g_i$. Invariants are given by the traces of covariants. The operators $\partial_1 + h_{11}$ and $\partial_2 + h_{22}$ are of types $G_1 \times G_1$ and $G_2 \times G_2$ respectively:

$$\partial_1 + h'_{11} = g_1^{-1}(\partial_1 + h_{11})g_1,$$
$$\partial_2 + h'_{22} = g_2^{-1}(\partial_2 + h_{22})g_2.$$

But they are *differential operator covariants*. We seek *matrix covariants*. The simplest matrix covariants are $h_{12}h_{21}$ of type $G_1 \times G_1$ and $h_{21}h_{12}$ of type $G_2 \times G_2$ since,

$$h'_{12}h'_{21} = g_1^{-1}(h_{12}h_{21})g_1,$$
$$h'_{21}h'_{12} = g_2^{-1}(h_{21}h_{12})g_2.$$

Let us call $(\mathbf{12}) = h_{12}h_{21}$ and $(\mathbf{21}) = h_{21}h_{12}$, where $(\mathbf{12}) \in M_{m_1m_1}$ and $(\mathbf{21}) \in M_{m_2m_2}$. Note that we use similar notation as before but now $(\mathbf{12}) \neq (\mathbf{21})$. Our aim is now to form higher matrix covariants. For simplicity we call $\partial_1 + h_{11} = D_1$ and $\partial_2 + h_{22} = D_2$. The operators D_1 and D_2 are of type $G_1 \times G_1$ and $G_2 \times G_2$ respectively. Therefore one easily see that

$$h_{21}D_1 \quad \text{and} \quad D_2h_{21} \quad \text{are of type } G_2 \times G_1,$$
$$h_{12}D_2 \quad \text{and} \quad D_1h_{12} \quad \text{are of type } G_1 \times G_2.$$

Hence

$$c_{11} = h_{12}D_2h_{21}D_1 - D_1h_{12}D_2h_{21} \quad \text{is of type } G_1 \times G_1,$$
$$c_{22} = h_{21}D_1h_{12}D_2 - D_2h_{21}D_1h_{12} \quad \text{is of type } G_2 \times G_2.$$

But these are still not matrix covariants, since they have leading differential operator terms

$$c_{11} = -[D_1, (\mathbf{12})]\partial_2 + \cdots,$$
$$c_{22} = -[D_2, (\mathbf{21})]\partial_1 + \cdots.$$

We would like to subtract off multiples of $\partial_2 + h_{22}$ from c_{11} and $\partial_1 + h_{11}$ from c_{22} to remove the differential operators but each operator is of the wrong type. To circumvent this we turn c_{11}, c_{22} into respectively $G_2 \times G_2$ and $G_1 \times G_1$ of type covariants by:

$$\begin{aligned} h_{21}c_{11}h_{12} &= -h_{21}[D_1, (\mathbf{12})]\partial_2 h_{12} + \cdots \text{matrix} \\ &= -h_{21}[D_1, (\mathbf{12})]h_{12}(\partial_2 + h_{22}) + \cdots \text{matrix}, \end{aligned} \tag{3.3}$$

$$h_{12}c_{22}h_{21} = -h_{12}[D_2, (\mathbf{21})]h_{21}(\partial_1 + h_{11}) + \cdots \text{matrix}. \tag{3.4}$$

Since each part in expression (3.3) is now of type $G_2 \times G_2$ and each part in (3.4) of type $G_1 \times G_1$, we must have matrix covariants:

$$[\mathbf{12}] = h_{12}c_{22}h_{21} + h_{12}[D_2, (\mathbf{21})]h_{21}(\partial_1 + h_{11}),$$
$$[\mathbf{21}] = h_{21}c_{11}h_{12} + h_{21}[D_1, (\mathbf{12})]h_{12}(\partial_2 + h_{22}).$$

Simplifying these give

$$[\mathbf{12}] = (\mathbf{12})[D_1, h_{12}D_2h_{21}] - h_{12}D_2h_{21}[D_1, (\mathbf{12})] \quad \text{of type } G_1 \times G_1, \tag{3.5}$$
$$[\mathbf{21}] = (\mathbf{21})[D_2, h_{21}D_1h_{12}] - h_{21}D_1h_{12}[D_2, (\mathbf{21})] \quad \text{of type } G_2 \times G_2 \tag{3.6}$$

as matrix covariants where $[\mathbf{12}] \in M_{m_1m_1}$ and $[\mathbf{21}] \in M_{m_2m_2}$.

The case $m_1 = m_2 = 1$:

We find a reduction of $[\mathbf{12}]$ and $[\mathbf{21}]$ in the case $m_1 = m_2 = 1$. So in this case $(\mathbf{12}) = h_{12}h_{21}$ and $(\mathbf{21}) = h_{21}h_{12}$ are just equal functions and $(\mathbf{21}) = (\mathbf{12}) = (12)$, the earlier invariant. By substituting $D_1 = \partial_1 + h_{11}$ and $D_2 = \partial_2 + h_{22}$ in the covariants $[\mathbf{12}]$, $[\mathbf{21}]$ and then by doing some differential and algebraic calculations we obtain the function covariants as follows:

$$\begin{aligned} [\mathbf{12}] &= -\frac{1}{4}(12)^2_{,12} - (12)^2[12], \\ [\mathbf{21}] &= -\frac{1}{4}(12)^2_{,12} + (12)^2[12]. \end{aligned} \tag{3.7}$$

One easily sees that

$$\begin{aligned} [\mathbf{12}] + [\mathbf{21}] &= -\frac{1}{2}(12)^2_{,12}, \\ [\mathbf{21}] - [\mathbf{12}] &= 2(12)^2[12]. \end{aligned} \tag{3.8}$$

Thus relating the expressions from the new covariants to the old invariants in this case $(m_1 = m_2 = 1)$.

3.4. *The case where rank exceeds dimension*

It is clear that in the case where the rank r is larger than the dimension n we may attempt to repeat the arguments of Sec. 2 under the weaker hypothesis that the h_{ij} and g_i are matrices and no longer (commuting) functions. The canonical form (2.4) still suffices where now the h_{ij} are rectangular matrices of type $m_i \times m_j$, where an $m_i \times m_i$ unit matrix is taken to stand (but omitted) before each operator, ∂_i, and where $m_1 + m_2 + \cdots + m_n = r$.

The case $n = 2$:

In this case we consider a differential matrix operator $\mathbb{L}$ such that

$$\mathbb{L} = \begin{pmatrix} \partial_1 & h_{12} \\ h_{21} & \partial_2 \end{pmatrix}, \tag{3.9}$$

where $h_{12} \in M_{m_1m_2}$ and $h_{21} \in M_{m_2m_1}$ are matrix functions of x_1 and x_2. We have assumed a gauge transformation to this form as before.

The gauge transformation

$$\mathbb{L} \mapsto \mathbb{L}' = g^{-1}\mathbb{L}g, \tag{3.10}$$

where

$$g = \begin{pmatrix} g_1(x_2) & 0 \\ 0 & g_2(x_1) \end{pmatrix}, \tag{3.11}$$

gives us

$$h_{12} \mapsto h'_{12} = g_1^{-1}h_{12}g_2, \tag{3.12}$$
$$h_{21} \mapsto h'_{21} = g_2^{-1}h_{21}g_1, \tag{3.13}$$

where g_1 and g_2 are invertible square matrices such that $g_1 \in M_{m_1m_1}$ and $g_2 \in M_{m_2m_2}$. So the relations (3.12) and (3.13) give us

$$h'_{12}h'_{21} = g_1^{-1}(h_{12}h_{21})g_1,$$
$$h'_{21}h'_{12} = g_2^{-1}(h_{21}h_{12})g_2.$$

Thus we have

$$(\mathbf{12})' = g_1^{-1}(\mathbf{12})g_1, \tag{3.14}$$
$$(\mathbf{21})' = g_2^{-1}(\mathbf{21})g_2, \tag{3.15}$$

where $(\mathbf{12}) \in M_{m_1m_1}$ and $(\mathbf{21}) \in M_{m_2m_2}$ are matrix covariants such that

$$(\mathbf{12}) = h_{12}h_{21}, \tag{3.16}$$
$$(\mathbf{21}) = h_{21}h_{12}. \tag{3.17}$$

By doing some algebraic calculations over (3.12) and (3.13) we obtain

$$(\mathbf{12})'(h'_{12,2}h'_{21})_{,1} + h'_{12}h'_{21,2}(\mathbf{12})'_{,1} = g_1^{-1}((\mathbf{12})(h_{12,2}h_{21})_{,1} + h_{12}h_{21,2}(\mathbf{12})_{,1})g_1,$$
$$(\mathbf{21})'(h'_{21,1}h'_{12})_{,2} + h'_{21}h'_{12,1}(\mathbf{21})'_{,2} = g_2^{-1}((\mathbf{21})(h_{21,1}h_{12})_{,2} + h_{21}h_{12,1}(\mathbf{21})_{,2})g_2.$$

Therefore we have

$$[\mathbf{12}]' = g_1^{-1}[\mathbf{12}]g_1, \tag{3.18}$$
$$[\mathbf{21}]' = g_2^{-1}[\mathbf{21}]g_2, \tag{3.19}$$

where we define matrix covariants $[\mathbf{12}] \in M_{m_1m_1}$ and $[\mathbf{21}] \in M_{m_2m_2}$ as follows

$$[\mathbf{12}] = (\mathbf{12})(h_{12,2}h_{21})_{,1} + h_{12}h_{21,2}(\mathbf{12})_{,1}, \tag{3.20}$$
$$[\mathbf{21}] = (\mathbf{21})(h_{21,1}h_{12})_{,2} + h_{21}h_{12,1}(\mathbf{21})_{,2}. \tag{3.21}$$

Before we move to the case $n = 3$, we compare our covariants $(\mathbf{12}), (\mathbf{21}), [\mathbf{12}], [\mathbf{21}]$ with Konopelchenko's covariants (3.2) [9]: $h = a_{,x} + ba - c$, $k = b_{,y} + ab - c$, where h and k are covariants for the hyperbolic system $z_{xy} + az_x + bz_y + cz = 0$. This corresponds to $m_1 = m_2$ in the current context. As we already know this system can be written in a differential operator form as $Lz = (\partial_x\partial_y + a\partial_x + b\partial_y + c)z = 0$, where the differential operator $L = \partial_x\partial_y + a\partial_x + b\partial_y + c$ can be written as

$$\begin{aligned} L &= (\partial_x + b)(\partial_y + a) - h \\ &= (\partial_y + a)(\partial_x + b) - k. \end{aligned}$$

Therefore, we can rewrite the above system $Lz = 0$ as

$$Au = 0, \tag{3.22}$$

$$Bv = 0, \tag{3.23}$$

where

$$A = \begin{pmatrix} \partial_x + b & -h \\ -I & \partial_y + a \end{pmatrix}, \quad u = \begin{pmatrix} z_1 \\ z \end{pmatrix}; \quad B = \begin{pmatrix} \partial_x + b & -I \\ -k & \partial_y + a \end{pmatrix}, \quad v = \begin{pmatrix} z \\ z_2 \end{pmatrix}.$$

For the system (3.22), we obtain covariant relations:

$$\begin{aligned} (\mathbf{12}) &= (\mathbf{21}) = h, \\ [\mathbf{12}] &= hh_{xy}, \\ [\mathbf{21}] &= h_x h_y, \end{aligned} \tag{3.24}$$

where $m_1 = m_2 = m$ and $\partial_1 = \partial_x,\ \partial_2 = \partial_y$.

We can easily see that

$$[\mathbf{12}] + [\mathbf{21}] = \frac{1}{2}(hh_y)_x. \tag{3.25}$$

Thus, we have

$$\mathrm{Tr}[\mathbf{12}] + \mathrm{Tr}[\mathbf{21}] = \frac{1}{2}[\mathrm{Tr}(\mathbf{12})(\mathbf{21})]_{xy}. \tag{3.26}$$

Similarly, for the system (3.23), we have the following relations:

$$\begin{aligned} (\mathbf{12}) &= (\mathbf{21}) = k, \\ [\mathbf{12}] &= k_y k_x, \\ [\mathbf{21}] &= kk_{xy}. \end{aligned} \tag{3.27}$$

These relations give us

$$[\mathbf{12}] + [\mathbf{21}] = \frac{1}{2}(kk_x)_y. \tag{3.28}$$

Once again, we have

$$\mathrm{Tr}[\mathbf{12}] + \mathrm{Tr}[\mathbf{21}] = \frac{1}{2}[\mathrm{Tr}(\mathbf{12})(\mathbf{21})]_{xy}. \tag{3.29}$$

The case $n = 3$:

Here we consider a differential matrix operator $\mathbb{L}$ such that

$$\mathbb{L} = \begin{pmatrix} \partial_1 & h_{12} & h_{13} \\ h_{21} & \partial_2 & h_{23} \\ h_{31} & h_{32} & \partial_3 \end{pmatrix}, \tag{3.30}$$

where $h_{ij} \in M_{m_i m_j}$ $(i, j = 1, 2, 3)$ are functions of x_1, x_2 and x_3.

Applying the gauge transformation

$$\mathbb{L} \mapsto \mathbb{L}' = g^{-1}\mathbb{L}g, \tag{3.31}$$

where

$$g = \begin{pmatrix} g_1(x_2, x_3) & 0 & 0 \\ 0 & g_2(x_1, x_3) & 0 \\ 0 & 0 & g_3(x_1, x_2) \end{pmatrix}, \tag{3.32}$$

gives us

$$h'_{12} = g_1^{-1}h_{12}g_2, \quad h'_{13} = g_1^{-1}h_{13}g_3, \tag{3.33}$$

$$h'_{21} = g_2^{-1}h_{21}g_1, \quad h'_{23} = g_2^{-1}h_{23}g_3, \tag{3.34}$$

$$h'_{31} = g_3^{-1}h_{31}g_1, \quad h'_{32} = g_3^{-1}h_{32}g_2, \tag{3.35}$$

where g_1, g_2 and g_3 are invertible square matrices such that $g_1 \in M_{m_1m_1}$, $g_2 \in M_{m_2m_2}$ and $g_3 \in M_{m_3m_3}$.

By doing some algebraic calculation over the above relations (3.33)–(3.35), we obtain the following matrix covariants:

$$(\mathbf{12}) = h_{12}h_{21}, \quad (\mathbf{13}) = h_{13}h_{31}, \tag{3.36}$$

$$(\mathbf{23}) = h_{23}h_{32}, \quad (\mathbf{21}) = h_{21}h_{12}, \tag{3.37}$$

$$(\mathbf{31}) = h_{31}h_{13}, \quad (\mathbf{32}) = h_{32}h_{23}, \tag{3.38}$$

$$(\mathbf{123}) = h_{12}h_{23}h_{31}, \quad (\mathbf{132}) = h_{13}h_{32}h_{21}, \tag{3.39}$$

$$(\mathbf{231}) = h_{23}h_{31}h_{12}, \quad (\mathbf{213}) = h_{21}h_{13}h_{32}, \tag{3.40}$$

$$(\mathbf{312}) = h_{31}h_{12}h_{23}, \quad (\mathbf{321}) = h_{32}h_{21}h_{13}, \tag{3.41}$$

$$[\mathbf{12}] = (\mathbf{12})(h_{12,2}h_{21})_{,1} + h_{12}h_{21,2}(\mathbf{12})_{,1}, \tag{3.42}$$

$$[\mathbf{13}] = (\mathbf{13})(h_{13,3}h_{31})_{,1} + h_{13}h_{31,3}(\mathbf{13})_{,1}, \tag{3.43}$$

$$[\mathbf{21}] = (\mathbf{21})(h_{21,1}h_{12})_{,2} + h_{21}h_{12,1}(\mathbf{21})_{,2}, \tag{3.44}$$

$$[\mathbf{23}] = (\mathbf{23})(h_{23,3}h_{32})_{,2} + h_{23}h_{32,3}(\mathbf{23})_{,2}, \tag{3.45}$$

$$[\mathbf{31}] = (\mathbf{31})(h_{31,1}h_{13})_{,3} + h_{31}h_{13,1}(\mathbf{31})_{,3}, \tag{3.46}$$

$$[\mathbf{32}] = (\mathbf{32})(h_{32,2}h_{23})_{,3} + h_{32}h_{23,2}(\mathbf{32})_{,3}, \tag{3.47}$$

where $(\boldsymbol{ij}), (\boldsymbol{ijk}), [\boldsymbol{ij}] \in M_{m_i m_i}(i, j, k \in \{1, 2, 3\})$.

The question of functional relations between covariants is more subtle than for invariants.

The general case:

Let us consider the following differential operator

$$\mathbb{L} = \begin{pmatrix} I_{m_1}\partial_1 & h_{12} & \cdots & h_{1n} \\ h_{21} & I_{m_2}\partial_2 & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \cdots & I_{m_n}\partial_n \end{pmatrix},$$

where the h_{ij} are functions of $x_1, x_2, \ldots, x_n$ and the I_{m_i} are unit matrices such that $h_{ij} \in M_{m_i m_j}$ and $I_{m_i} \in M_{m_i m_i}$ where $i, j \in \{1, 2, \ldots, n\}$.

The gauge transformation

$$\mathbb{L} \mapsto \mathbb{L}' = g^{-1}\mathbb{L}g, \tag{3.48}$$

where

$$g = \begin{pmatrix} g_1(x_2, x_3, \ldots, x_n) & 0 & \cdots & 0 \\ 0 & g_2(x_1, x_3, \ldots, x_n) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & g_n(x_1, x_2, \ldots, x_{n-1}) \end{pmatrix}, \tag{3.49}$$

gives us

$$h_{ij} \mapsto h'_{ij} = g_i^{-1} h_{ij} g_j, \tag{3.50}$$

where the g_i are square matrices such that $g_i \in M_{m_i m_i}$.

The relations (3.50) gives us the following matrix covariants:

$$[\boldsymbol{ij}] = (\boldsymbol{ij})(h_{ij,j}h_{ji})_{,i} + h_{ij}h_{ji,j}(\boldsymbol{ij})_{,i} \tag{3.51}$$

$$(\boldsymbol{i_1 i_2 i_3 \cdots i_n}) = h_{i_1 i_2} h_{i_2 i_3} \cdots h_{i_n i_1} \tag{3.52}$$

where $[\boldsymbol{ij}] \in M_{m_i m_i}$ and $(\boldsymbol{i_1 i_2 i_3 \cdots i_n}) \in M_{m_{i_1} m_{i_1}}$.

4. Conclusions and Comments

In this paper, we have dealt with general hyperbolic systems $\mathbb{L}z = 0$. We have used a suitable diagonal gauge matrix g, chosen so that it kills diagonal terms h_{ii} where $i = 1, 2, \ldots, n$. We have also obtained the complete set of invariants for general hyperbolic systems where rank equals dimension by using the gauge transformation $\mathbb{L} \mapsto \mathbb{L}' = g^{-1}\mathbb{L}g$. Further, we have shown the completeness of a set of simple invariants (reduced invariants). We have proved that these invariants form a minimal complete set.

We have also considered hyperbolic systems $\mathbb{L}z = 0$ where the entries h_{ij} are matrices. In this case, we are interested in covariants. We have obtained matrix covariants for the differential operator $\mathbb{L}$ under the gauge transformation. Here we have examined the case where rank exceeds dimension. The canonical form of $\mathbb{L}$ still suffices where $h_{ii} = 0$ and h_{ij} are rectangular matrices. The reduced covariants have been presented but it has not been shown that their invariant traces form a complete set. For example, in the case when $n = 2$, we ask the question: Do the covariants $(\mathbf{12})$, $(\mathbf{21})$, $[\mathbf{12}]$ and $[\mathbf{21}]$ form a complete set? The answer depends on the existence of $g(x_1, x_2)$ so that when

$$\begin{aligned}(\mathbf{12})' &= g_1^{-1}(\mathbf{12})g_1,\\ (\mathbf{21})' &= g_2^{-1}(\mathbf{21})g_2,\\ [\mathbf{12}]' &= g_1^{-1}[\mathbf{12}]g_1,\\ [\mathbf{21}]' &= g_2^{-1}[\mathbf{21}]g_2\end{aligned} \tag{4.1}$$

are given then g must satisfy the relation

$$g^{-1}\mathbb{L}g = \mathbb{L}'.$$

The square matrices $(\boldsymbol{ij})$, $(\boldsymbol{ij})'$, $[\boldsymbol{ij}]$, $[\boldsymbol{ij}]'$ are thus similar to (4.1) and so possess as equal invariants the traces, say, of their powers: $I_p = \mathrm{Tr}(\boldsymbol{ij})^p$ etc. But equality of such invariants is not sufficient for gauge equivalence of $\mathbb{L}'$ and $\mathbb{L}$. There are also invariants associated with polynomials in $(\boldsymbol{ij})$ and $[\boldsymbol{ij}]$ since $g_i(\boldsymbol{ij})[\boldsymbol{ij}] = (\boldsymbol{ij})'[\boldsymbol{ij}]'g_i$ etc., namely, traces of such polynomials (cf. (3.26)).

Two questions arise for further study:

(1) What relations on the invariants of these general systems correspond to specializations of $\mathbb{L}$ such as self-adjointness?
(2) Can we establish the existence of a complete, minimal set of trace polynomial invariants for the systems of Sec. 3?

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