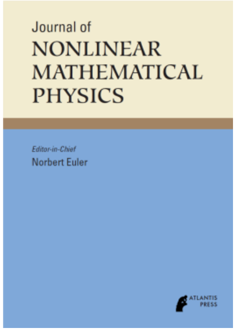

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NEW SOLVABLE MANY-BODY MODEL OF GOLDFISH TYPE

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A new *solvable* N-body model of goldfish type is identified. Its Newtonian equations of motion read as follows:

$$\ddot{z}_n = -6\dot{z}_n z_n - 4z_n^3 + \frac{3}{2}(\dot{z}_n + 2z_n^2)\sum_{k=1}^{N}\left(\frac{\dot{z}_k}{z_k} + 2z_k\right) + 2\sum_{\ell=1,\ell\neq n}^{N}\left[\frac{(\dot{z}_n + 2z_n^2)(\dot{z}_\ell + 2z_\ell^2)}{z_n - z_\ell}\right], \quad n = 1, \ldots, N,$$

where $z_n \equiv z_n(t)$ are the N dependent variables (with N an arbitrary positive integer), t is the independent variable ("time") and the dots indicate time-differentiations. Its *isochronous* variant is also obtained and discussed. Other new *solvable* models of goldfish type characterize the behavior of these systems in the immediate neighborhood of their equilibria.

Keywords: Integrable dynamical systems; solvable dynamical systems; integrable Newtonian many-body problems; solvable Newtonian many-body problems; isochronous dynamical systems.

Mathematical Subject Classification: 70F10, 70H06, 37J35, 37K10

1. Introduction

The basic goldfish dynamical system [1] — also interpretable as the simplest dynamical system belonging to the integrable Ruijsenaars–Schneider class [9] — is defined by the following Newtonian equations of motion:

$$\ddot{z}_n = 2\sum_{\ell=1,\ell\neq n}^{N}\left(\frac{\dot{z}_n\dot{z}_\ell}{z_n - z_\ell}\right), \quad n = 1, \ldots, N. \tag{1a}$$

Notation 1.1. Here and hereafter N is an arbitrary positive integer, indices such as n, ℓ run from 1 to N unless otherwise indicated (but for the convenience of the reader we generally specify below the range of the indices), t ("time") is the independent (*real*) variable, the

N coordinates z_n are the dependent variables, $z_n \equiv z_n(t)$, and superimposed dots denote time-differentiations.

It is generally convenient to assume that the dependent variables z_n are *complex* numbers, identifying the positions of N points moving in the *complex* z-plane. By introducing the *real* and *imaginary* parts of the coordinates $z_n = x_n + iy_n$ and correspondingly the *real* two-vectors $\vec{r}_n = (x_n, y_n)$, to the motions in the *complex* z-plane of points identified by the *complex* coordinates z_n there correspond the motions of *real* points whose positions in the *horizontal* plane are identified by the *real* two-vectors $\vec{r}_n$; this provides an added justification for referring to dynamical systems of goldfish type as *many-body* problems, as we occasionally do (also in the title of this paper). Moreover, it is often possible to write the corresponding equations of motion in *covariant* form. It would be possible to do so also for the many-body models considered in this paper, but we refrain from doing so; readers interested in pursuing this transformation are referred to Chap. 4 (entitled "Solvable and/or integrable many-body problems in the plane, obtained by complexification") of [2].

For an explanation of the origin of the term "goldfish" see [3; 4, p. 7].

A remarkable property of the many-body problem (1a) is its *solvability*, given by this neat prescription: the N values of the N dependent variables at time $t, z_n(t)$, are the N roots of the following algebraic equation for the unknown z,

$$\sum_{n=1}^{N} \left[\frac{\dot{z}_n(0)}{z - z_n(0)} \right] = \frac{1}{t}, \tag{1b}$$

where $\dot{z}_n(0)$ respectively $z_n(0)$ are the *initial* velocities respectively positions of the N point-like moving particles [1, 4, 2, 3]. Note that this equation, after multiplication by the N-degree polynomial $\prod_{m=1}^{N}[z - z_m(0)]$, becomes a (time-dependent) polynomial equation of degree N for the unknown z, demonstrating thereby that it indeed yields N values $z_n \equiv z_n(t)$ as its solutions.

Over time several more N-body models "of goldfish type" have been identified, their signature being provided by the presence in the right-hand side of the Newtonian equations ("accelerations equal forces") characterizing them — in addition to other contributions — of a term identical to that appearing in the right-hand side of (1a). All these models are *solvable*, meaning that their initial-value problem can be solved by *algebraic* operations, generally by finding the N eigenvalues of an explicitly known time-dependent $N \times N$ matrix, or equivalently, as in the case described above, see (1b), by finding the N roots of a time-dependent polynomial of degree N: see [2] and especially [4, Sec. 4.2.2]. Recent additions to this catalog of *solvable* many-body problems of goldfish type are reported in [5–7]. Purpose and scope of the present paper is to provide one more — indeed, see below, a few — new *solvable* many-body models of goldfish type. It is quite possible that additional *new* examples will be found over time: we consider such a search worthwhile, and any such finding remarkable.

The basic new model, and its solution, are described in the following Sec. 2, firstly in its *nonisochronous* version, and then in its (more general) *isochronous* variant. Other versions of these models are also obtained. The equilibrium configurations of these models are also reported, and new, nontrivial, *solvable* models of goldfish type characterizing the behavior in the immediate neighborhood of these equilibria are exhibited. Proofs are then provided in Sec. 3.

2. Results

In this section, we report the main results, firstly for the new model and then for its *isochronous* variant.

2.1. *The nonisochronous model*

The *new* model is defined by the following Newtonian equations of motion:

$$\ddot{z}_n = \frac{3}{2}\frac{\dot{z}_n^2}{z_n} + 2z_n^3 + (\dot{z}_n + 2z_n^2)\sum_{\ell=1,\ell\neq n}^{N}\left[(\dot{z}_\ell + 2z_\ell^2)\left(\frac{3}{2z_\ell} + \frac{2}{z_n - z_\ell}\right)\right], \quad n = 1,\ldots,N, \tag{2a}$$

or, equivalently,

$$\ddot{z}_n = -6\dot{z}_n z_n - 4z_n^3 + \frac{3}{2}(\dot{z}_n + 2z_n^2)\eta + 2\sum_{\ell=1,\ell\neq n}^{N}\left[\frac{(\dot{z}_n + 2z_n^2)(\dot{z}_\ell + 2z_\ell^2)}{z_n - z_\ell}\right], \quad n = 1,\ldots,N, \tag{2b}$$

where

$$\eta = \sum_{k=1}^{N}\left(\frac{\dot{z}_k}{z_k} + 2z_k\right). \tag{2c}$$

The solution of the corresponding initial-value problem is given by the following

Proposition 2.1. *The N coordinates $z_n(t)$ are the N eigenvalues of the following $N \times N$ matrix $U(t)$,*

$$U(t) = [1 - tY(0)]^{-1}Z(0)\cdot\{1 - tY(0) - t^2 Z(0)[1 - tY(0)]^{-1}Z(0)\}^{-1} \tag{3a}$$

with the two $N \times N$ matrices $Z(0)$ and $Y(0)$ expressed componentwise as follows in terms of the initial data $z_n(0), \dot{z}_n(0)$:

$$Z(0) = \mathrm{diag}[z_n(0)], \quad Z_{nm}(0) = \delta_{nm} z_n(0), \quad n, m = 1,\ldots,N, \tag{3b}$$

$$Y_{nm}(0) = \delta_{nm}\frac{\dot{z}_n(0)}{2z_n(0)} + (1 - \delta_{nm})\frac{1}{2}\left[\frac{\dot{z}_n(0)}{z_n(0)} + 2z_n(0)\right]^{1/2}\left[\frac{\dot{z}_m(0)}{z_m(0)} + 2z_m(0)\right]^{1/2}, \quad n, m = 1,\ldots,N. \tag{3c}$$

Here and hereafter δ_{nm} is the standard Kronecker symbol, $\delta_{nm} = 1$ if $n = m, \delta_{nm} = 0$ if $n \neq m$.

It is thus seen that the solution of the initial-value problem for the Newtonian equations of motion (2) characterizing our new N-body problem of goldfish type is achievable by purely algebraic operations, namely by finding the N eigenvalues $z_n(t)$ of the $N \times N$ matrix

$U(t)$ given, in terms of the $2N$ initial values $z_m(0), \dot{z}_m(0), m = 1, \ldots, N$, and of the time t, by the explicit (if complicated) formulas (3).

Remark 2.1. For $N = 1$, the system (2) reduces to the single equation of motion (with $z_1(t) = z(t)$)

$$\ddot{z} = \frac{3}{2}\frac{\dot{z}^2}{z} + 2z^3 \tag{4a}$$

and the solution of the corresponding initial-value problem reads

$$z(t) = \frac{z(0)}{1 - 2vt + [v^2 - z^2(0)]t^2}, \quad v = \frac{\dot{z}(0)}{2z(0)}. \tag{4b}$$

Clearly in this case the only equilibrium configuration is $z = \dot{z} = 0$.

Remark 2.2. The asymptotic behavior as $t \to \infty$ of the $N \times N$ matrix $U(t)$, see (3), reads

$$U(t) = -t^{-2}\{Z(0) - Y(0)[Z(0)]^{-1}Y(0)\}^{-1}[1 + O(t^{-1})], \tag{5}$$

entailing that all the coordinates $z_n(t)$ vanish asymptotically proportionally to t^{-2}.

Remark 2.3. It is easily seen, by summing the N ODEs (2b) divided by z_n, by using the definition (2c) of η, by taking advantage of the possibility to exchange the role of the two dummy indices n and ℓ when both are summed upon, and by utilizing the identity

$$\frac{1}{z_n - z_\ell}\left(\frac{1}{z_n} - \frac{1}{z_\ell}\right) = -\frac{1}{z_n}\frac{1}{z_\ell}, \tag{6}$$

that one gets the ODE

$$\dot{\eta} = \frac{\eta^2}{2} \tag{7}$$

yielding

$$\eta(t) = \frac{2\eta(0)}{1 - \eta(0)t} = -\frac{2}{t + t_0}, \tag{8a}$$

of course with

$$t_0 = -\frac{2}{\eta(0)} = -2\left\{\sum_{k=1}^{N}\left[\frac{\dot{z}_k(0)}{z_k(0)} + 2z_k(0)\right]\right\}^{-1}. \tag{8b}$$

Hence an equivalent, *nonautonomous*, version of the many-body model (2b) obtains by replacing in these equations of motion the expression (2c) of the collective coordinate η in terms of z_n and $\dot{z}_n$ with its explicit time-dependent expression (8).

Proposition 2.2. *For this many-body model with $N \geq 2$ the equilibrium configuration — defined of course up to permutations of the N coordinates — features $N - 2$ vanishing coordinates and the remaining 2 having equal and opposite values, say $z_n = \dot{z}_n = 0, n = 1, \ldots, N - 2$ and $z_{N-1} = \bar{z}, z_N = -\bar{z}$ with $\bar{z}$ an arbitrary constant* (*possibly also vanishing*); *of course to the extent the last term in the right-hand side of* (2a) *is assigned a vanishing value when $z_n = \dot{z}_n = 0$ and $z_\ell = \dot{z}_\ell = 0$.*

The behavior of the *nonisochronous* system (2b) in the immediate neighborhood of its equilibrium configuration is described by an, of course *solvable*, dynamical system of goldfish type. This is obtained from (2b) by setting, say,

$$z_{N-1}(t) = \bar{z} + \varepsilon y_1(t); \quad z_N(t) = -\bar{z} + \varepsilon y_2(t);$$
$$z_n(t) = \varepsilon x_n(t), \quad n = 1, \ldots, N_0 = N - 2, \tag{9}$$

with $\bar{z}$ an arbitrary constant and ε infinitesimal. Here we assume the constant $\bar{z}$ to be finite (nonvanishing): the analogous treatments of the cases when $\bar{z}$ vanishes or is itself of order ε are left for the alert reader. The relevant equations are conveniently written as follows:

$$\ddot{f} = \frac{3}{2}\chi\dot{f} + 8\bar{z}^2 f + 6\bar{z}\chi g + 12\bar{z}^2\varphi, \tag{10a}$$

$$\ddot{g} = \frac{3}{2}\chi\dot{g} - 4\bar{z}^2 g - 2\bar{z}\dot{f} + 6\bar{z}\chi f + 8\bar{z}\dot{\varphi}, \tag{10b}$$

$$\ddot{x}_n = \frac{3}{2}\chi\dot{x}_n + 2\sum_{\ell=1,\ell\neq n}^{N_0}\left(\frac{\dot{x}_n\dot{x}_\ell}{x_n - x_\ell}\right), \quad n = 1, \ldots, N_0, \tag{10c}$$

where

$$f = y_1 + y_2, \quad g = y_1 - y_2; \quad \varphi = \sum_{n=1}^{N_0}(x_n), \quad \chi = \sum_{n=1}^{N_0}\left(\frac{\dot{x}_n}{x_n}\right). \tag{10d}$$

Note that the system (10c) with the collective coordinate χ defined by (10d) and N_0 an arbitrary positive integer is an (of course *solvable*) dynamical system of goldfish type (not new, see [4, Eq. (4.135)] with $\alpha = b = 0$ and $a = 1/2$).

Remark 2.4. It is easily seen — as in Remark 2.3 — that the system (10c) implies for the collective coordinate χ, see (10d), the ODE

$$\dot{\chi} = \frac{\chi^2}{2} \tag{11}$$

yielding

$$\chi(t) = \frac{2\chi(0)}{2 - \chi(0)t} = -\frac{2}{t + \bar{t}}, \tag{12a}$$

of course with

$$\bar{t} = -\frac{2}{\chi(0)} = -2\left\{\sum_{k=1}^{N}\left[\frac{\dot{x}_k(0)}{x_k(0)}\right]\right\}^{-1}. \tag{12b}$$

By inserting this expression, (12), of χ in (10c) one obtains an equivalent, *nonautonomous*, version of this dynamical system of goldfish type.

Likewise, by summing the N_0 ODEs (10c) and using the definition of φ, see (10d), one gets the ODE

$$\ddot{\varphi} = \frac{3}{2}\chi\dot{\varphi}, \tag{13}$$

which, using (12), is easily integrated, yielding

$$\varphi(t) = \varphi(0) + \frac{\dot\varphi(0)}{\chi(0)}\left\{1 - \frac{4}{[2-\chi(0)t]^2}\right\}$$
$$= \varphi(0) + \frac{\dot\varphi(0)\bar t}{2}\left[\frac{t(t+2\bar t\,)}{(t+\bar t\,)^2}\right]. \tag{14}$$

Hence by inserting these explicit expressions, (12) and (14), of the collective quantities $\chi(t)$ and $\varphi(t)$ in the two linear ODEs (10a) and (10b) — with $\chi(0)$ or $\bar t$ now playing the role of *arbitrary* constants — one obtains a *nonautonomous solvable* system of two linear second-order ODEs; or equivalently — by solving (10a) for g and inserting the resulting expression of g in (10b) — one obtains the following fourth-order linear nonautonomous single ODE for the dependent variable $f(t)$:

$$\ddot{\ddot f} = -\frac{8\dddot f}{t+\bar t} + 4\left(\bar z^2 - \frac{3}{(t+\bar t)^2}\right)\ddot f + \frac{48\bar z^2}{t+\bar t}\dot f + \frac{\bar z^2}{2}\left(64\bar z^2 + \frac{21}{(t+\bar t)^2}\right)f + \bar z^2 h, \tag{15a}$$

with

$$h(t) = \left(48\bar z^2 + \frac{9}{4(t+\bar t)^2}\right)\varphi(0)$$
$$+\left[24\bar z^2 + \left(\frac{9}{8} - 24\bar z^2\bar t^2\right)\frac{1}{(t+\bar t)^2} - \frac{273}{8}\frac{\bar t^2}{(t+\bar t\,)^4}\right]\bar t\dot\varphi(0), \tag{15b}$$

where $\bar z, \bar t, \varphi(0)$ and $\dot\varphi(0)$ are 4 arbitrary constants. The possibility is far from trivial — yet implied by the above developments — to obtain by algebraic operations the solution of the initial-value problem for this ODE, (15), corresponding to an arbitrary assignment of the 4 initial data $f(0)$, $\dot f(0)$, $\ddot f(0)$, $\dddot f(0)$, or equivalently of the initial-value problem for the system of two coupled ODEs (10a) and (10b) with (12) and (14) corresponding to an arbitrary assignment of the 4 initial data $f(0)$, $\dot f(0)$, $g(0)$, $\dot g(0)$.

It is now convenient to introduce the (monic) polynomial $\psi(z,t)$ that features the N coordinates $z_n(t)$, $n = 1,\ldots,N$, as its N zeros and the N quantities $c_m(t), m = 1,\ldots,N$, as its N coefficients,

$$\psi(z,t) = \prod_{n=1}^{N}[z - z_n(t)] = z^N + \sum_{m=0}^{N} c_m(t)z^{N-m}. \tag{16a}$$

These formulas entail

$$\sum_{n=1}^{N}(z_n) = -c_1 \tag{16b}$$

and (see, if need be, [4, Eq. (A.11)])

$$\sum_{n=1}^{N}\left(\frac{\dot z_n}{z_n}\right) = \frac{\dot c_N}{c_N}. \tag{16c}$$

The definition (16a) implies a one-to-one (*nonlinear*) relation among the *unordered* set of the N zeros $z_n(t)$ and the *ordered* set of the N coefficients $c_m(t)$. Whenever the N coordinates $z_n(t)$ evolve according to a *solvable* dynamical system, the corresponding evolution of the N

coefficients $c_m(t)$ is as well *solvable*, since the relation among the N coordinates $z_n(t)$ and the N coefficients $c_m(t)$ entailed by the relation (16a) is obviously *algebraic*. It is moreover well known [1, 2, 4] that when the N coordinates $z_n(t)$ evolve according to a dynamical system of *goldfish* type, the corresponding dynamical system satisfied by the coordinates $c_m(t)$ is generally rather neat hence worth display (and its derivation amounts to a *standard* computation, facilitated by the identities reported in the [4, Appendix A]). We therefore now report, without detailing its derivation, the *solvable* dynamical system satisfied by the N coefficients $c_m(t)$ which corresponds to the *solvable* dynamical system of goldfish type (2):

$$\ddot{c}_m - \left(\frac{3\dot{c}_N}{2c_N} + c_1\right)\dot{c}_m + \left(\frac{3\dot{c}_N}{c_N}c_1 - 2c_1^2 - 2\dot{c}_1\right)c_m - \left[3(m+1)\frac{\dot{c}_N}{c_N} + 2(m-1)c_1\right]c_{m+1} + 2(2m+1)\dot{c}_{m+1} + 4m(m+2)c_{m+2} = 0, \quad c_{N+1} = c_{N+2} = 0, \quad m = 1, \ldots, N. \tag{17}$$

The initial data for this solvable dynamical system are of course the $2N$ numbers $c_m(0), \dot{c}_m(0), m = 1, \ldots, N$.

For $N = 1$ this system reduces of course to the single ODE (4a) (with $c_1(t) = -z(t)$).

And let us also display this *solvable* nonlinear system for $N = 2$:

$$\ddot{c}_1 - \frac{3\dot{c}_1\dot{c}_2}{2c_2} + \frac{3\dot{c}_2c_1^2}{c_2} - 3\dot{c}_1c_1 - 2c_1^3 = 0, \tag{18a}$$

$$\ddot{c}_2 - \frac{3\dot{c}_2^2}{2c_2} + 2\dot{c}_2c_1 - 2\dot{c}_1c_2 - 2c_1^2c_2 = 0. \tag{18b}$$

Remark 2.5. An equivalent, but *nonautonomous*, version of the equations of motion (17) obtains via the relation (implied by (2c) and (8) with (16b) and (16c))

$$\frac{\dot{c}_N(t)}{c_N(t)} = 2c_1(t) - \frac{2}{t+t_0}, \tag{19a}$$

of course now with

$$t_0 = -2\left[\frac{\dot{c}_N(0)}{c_N(0)} - 2c_1(0)\right]^{-1}. \tag{19b}$$

For instance the diligent reader will verify that, for $N = 2$, the insertion of the expression (19a) of $\dot{c}_2/c_2$ in (18b) yields an identity, while its insertion in (18a) yields the second-order *nonautonomous* ODE

$$\ddot{c}_1 = 6\dot{c}_1c_1 - 4c_1^3 - \frac{3(\dot{c}_1 - 2c_1^2)}{t+t_0}. \tag{20a}$$

In this ODE t_0 has become an *arbitrary* parameter, in fact quite irrelevant amounting simply to a shift of the independent variable t. It is easily seen that two special solutions of this nonlinear nonautonomous ODE read

$$c_1(t) = \pm\frac{1}{2(t+t_0)}. \tag{20b}$$

Again, less trivial is the possibility, entailed by the above developments, to obtain by elementary algebraic operations the solution of the initial-value problem for the ODE (20a) corresponding to an arbitrary assignment of the 2 initial data $c_1(0)$ and $\dot{c}_1(0)$.

2.2. *The isochronous model*

The *isochronous* model is obtained from the preceding one via the following standard trick (see, for instance [4, Sec. 2.1]). Let

$$\tau \equiv \tau(t) = \frac{\exp(i\omega t) - 1}{i\omega}. \tag{21a}$$

Here and throughout ω is an arbitrary *positive* constant, with the dimension of inverse time, to which we associate the period

$$T = \frac{2\pi}{\omega}. \tag{21b}$$

Introduce then the new dependent variables $\tilde{z}_n(t)$ by setting

$$\tilde{z}_n(t) = \exp(i\omega t) z_n(\tau), \quad z_n(t) = \exp(-i\omega t)\tilde{z}_n(\tau), \tag{22a}$$

so that

$$\dot{\tilde{z}}_n - i\omega\tilde{z}_n = \exp(2i\omega t) z_n'(\tau), \quad z_n'(\tau) = \exp(-2i\omega t)(\dot{\tilde{z}}_n - i\omega\tilde{z}_n), \tag{22b}$$

and

$$\ddot{\tilde{z}}_n - 3i\omega\dot{\tilde{z}}_n - 2\omega^2\tilde{z} = \exp(3i\omega t) z_n''(\tau). \tag{22c}$$

Here and above the appended primes denote of course differentiations with respect to the variable τ, $z_n'(\tau) \equiv dz_n(\tau)/d\tau$.

Note that by setting formally $\omega = 0$ one gets from (21a) $\tau = t$ and from (22a) $\tilde{z}_n(t) = z_n(t)$; hence in the $\omega \Rightarrow 0$ limit the results reported in this Sec. 2.2 reduce to those of the preceding Sec. 2.1 (whenever this limit makes sense).

The change of dependent variables (22) entails that the equations of motion (2b) get replaced by the following Newtonian equations of motion for the new dependent variables $\tilde{z}_n(t)$:

$$\begin{aligned}\ddot{\tilde{z}}_n = &-\frac{3}{2}(N-2)i\omega\dot{\tilde{z}}_n - \frac{1}{2}(3N-4)\omega^2\tilde{z}_n - 6\dot{\tilde{z}}_n\tilde{z}_n - 3(N-2)i\omega\tilde{z}_n^2 - 4\tilde{z}_n^3 \\ &+ \frac{3}{2}(\dot{\tilde{z}}_n - i\omega\tilde{z}_n + 2\tilde{z}_n^2)\tilde{\eta} + 2\sum_{\ell=1,\ell\neq n}^{N}\left[\frac{(\dot{\tilde{z}}_n - i\omega\tilde{z}_n + 2\tilde{z}_n^2)(\dot{\tilde{z}}_\ell - i\omega\tilde{z}_\ell + 2\tilde{z}_\ell^2)}{\tilde{z}_n - \tilde{z}_\ell}\right], \\ &\qquad n = 1, \ldots, N, \end{aligned} \tag{23a}$$

$$\tilde{\eta} = \sum_{k=1}^{N}\left(\frac{\dot{\tilde{z}}_k}{\tilde{z}_k} + 2\tilde{z}_k\right). \tag{23b}$$

And it is plain from Proposition 2.1 and the formulas (22) that the solution of the corresponding initial-value problem is provided by the following proposition.

Proposition 2.3. *The N coordinates $\tilde{z}_n(t)$ are the N eigenvalues of the following $N \times N$ matrix $\tilde{U}(t)$:*

$$\tilde{U}(t) = \exp(i\omega t)[1 - \tau(t)\tilde{Y}(0)]^{-1}\tilde{Z}(0) \cdot \{1 - \tau(t)\tilde{Y}(0) - \tau^2(t)\tilde{Z}(0)[1 - \tau(t)\tilde{Y}(0)]^{-1}\tilde{Z}(0)\}^{-1}, \tag{24a}$$

with $\tau(t)$ defined as above, see (21a), *and*

$$\tilde{Z}(0) = \mathrm{diag}[\tilde{z}_n(0)], \quad \tilde{Z}_{nm}(0) = \delta_{nm}\tilde{z}_n(0), \quad n, m = 1, \ldots, N, \tag{24b}$$

$$\tilde{Y}_{nm}(0) = \frac{1}{2}\delta_{nm}\left[\frac{\dot{\tilde{z}}_n(0)}{\tilde{z}_n(0)} - i\omega\right] + \frac{1}{2}(1 - \delta_{nm})\left[\frac{\dot{\tilde{z}}_n(0)}{\tilde{z}_n(0)} - i\omega + 2\tilde{z}_n(0)\right]^{1/2} \times \left[\frac{\dot{\tilde{z}}_m(0)}{\tilde{z}_m(0)} - i\omega + 2\tilde{z}_m(0)\right]^{1/2}, \quad n, m = 1, \ldots, N. \tag{24c}$$

Remark 2.6. Clearly (see (21)) the matrix $\tilde{U}(t)$, see (24a), evolves periodically,

$$\tilde{U}(t + T) = \tilde{U}(t). \tag{25a}$$

Hence the many-body problem characterized by the system of Newtonian equations of motion (23a) is *isochronous*, namely its *generic* solution evolves periodically with the fixed period T, or possibly with a period which is a, generally small, *integer* multiple of T: indeed, even though the $N \times N$ matrix $\tilde{U}(t)$ is itself periodic with period T, the time evolution of each of its N eigenvalues — which now takes *necessarily* place in the *complex* $\tilde{z}$-plane (except in the $N = 1$ case, see below) — might feature a somewhat larger periodicity due to the possibility that over the time evolution different eigenvalues exchange their roles (for a discussion of this phenomenology see [8]). For a *zero-measure* set of initial data the time evolution is *singular* due to the eventual occurrence of a collision of two (or, even more exceptionally, more than two) particles.

Remark 2.7. For $N = 1$, the system (23a) reduces to the following (*real*) equation of motion for the single dependent variable $\tilde{z}_1(t) = \tilde{z}(t)$, reading

$$\ddot{\tilde{z}} = \frac{3}{2}\frac{(\dot{\tilde{z}})^2}{\tilde{z}} + \frac{1}{2}\omega^2\tilde{z} + 2\tilde{z}^3; \tag{26a}$$

and the solution of the corresponding initial-value problem reads

$$\tilde{z}(t) = \frac{\tilde{z}(0)}{1 - a\sin(\omega t) - b\sin^2(\omega t/2)}, \quad a = \frac{\dot{\tilde{z}}(0)}{\omega\tilde{z}(0)}, \quad b = 1 + 4\left[\frac{\tilde{z}(0)}{\omega}\right]^2 + a^2. \tag{26b}$$

Clearly in this case there are three equilibria: $z(t) = \bar{z}^{(j)}$ with

$$\bar{z}^{(1)} = 0; \quad \bar{z}^{(2)} = \frac{i\omega}{2}; \quad \bar{z}^{(3)} = -\frac{i\omega}{2}. \tag{26c}$$

Remark 2.8. By a treatment analogous to that of Remark 2.3 one obtains from the system (23) the following ODE

$$2\dot{\tilde{\eta}} = [\tilde{\eta} + (N-1)i\omega]^2 + \omega^2, \tag{27}$$

yielding

$$\tilde{\eta}(t) = -\omega\left\{(N-1)i + \operatorname{cotg}\left[\frac{\omega(t+\tilde{t}_0)}{2}\right]\right\}, \tag{28a}$$

with $\tilde{t}_0$ defined of course by the following formula:

$$\operatorname{cotg}\left(\frac{\omega\tilde{t}_0}{2}\right) = -(N-1)i - \frac{\tilde{\eta}(0)}{\omega}. \tag{28b}$$

Hence an equivalent, *nonautonomous*, version of the many-body model (23) obtains by replacing in the equations of motion (23a) the expression (23b) of the collective coordinate $\tilde{\eta}$ in terms of $\tilde{z}_n$ and $\dot{\tilde{z}}_n$ with its explicit time-dependent expression (28).

In the *isochronous* case the phenomenology of the equilibria is a bit less simple than in the *nonisochronous* case (see above Proposition 2.2). To describe the situation it is convenient to rescale the equilibrium coordinates as follows:

$$\tilde{z}_n(t) = \bar{z}_n \equiv \left(\frac{i\omega}{2}\right)\zeta_n, \quad n = 1, \ldots, N. \tag{29}$$

For $N = 1$, the only 3 equilibria are those reported above (see Remark 2.7), corresponding to the 3 rescaled values $\zeta^{(1)} = 0$, $\zeta^{(2)} = 1$, $\zeta^{(3)} = -1$.

For $N \geq 2$, the equilibrium configurations (which are of course defined up to permutations) are detailed by the following

Proposition 2.4. *For $N \geq 2$, there are three types of equilibria of the system* (23).

The first type features (*rescaled*) *equilibrium coordinates ζ_n all of which take either one or the other of the two special values* 1 *or* 0, *say N_0 of them take the value* 0 *and $N_1 = N - N_0$ of them take the value* 1, *with N_0 an arbitrary non-negative integer not exceeding $N, 0 \leq N_0 \leq N$.*

The second type features N_0 (*rescaled*) *equilibrium coordinates taking the value* $0, N_1 = N - N_0 - 1$ *taking the special value* 1 (*of course now with* $0 \leq N_0 \leq N-1$) *and the remaining one taking the value $3N_0 - 1$.*

The third type features $N-2$ (*rescaled*) *equilibrium coordinates taking the value* 1 *and the remaining two coordinates taking the two values*

$$\zeta_1^{(1)} = \zeta, \quad \zeta_2^{(1)} = -\zeta; \quad \zeta_1^{(2)} = 1+\zeta, \quad \zeta_2^{(2)} = 1-\zeta, \tag{30}$$

with ζ an arbitrary number. Note that the solution of the third type also belongs to the first type (*with $N_0 = 2$*) *for $\zeta = 0$, and it also belongs to the second type* (*with $N_0 = 1$*) *for $\zeta = \pm 1$.*

The behavior of the *isochronous* system (23a) in the neighborhood of its equilibrium configuration is described by an, of course *solvable*, dynamical system of goldfish type which is itself interesting. We now exhibit the simplest instance corresponding to the *first type* of equilibria, see Proposition 2.4, leaving to the alert reader the derivation of the systems corresponding to the equilibria of *second* and *third types*. To this end we set

$$\tilde{z}_n(t) = \frac{i\omega}{2}\varepsilon\tilde{x}_n(t), \quad n = 1, \ldots, N_0, \tag{31a}$$

$$\tilde{z}_{N_0+n}(t) = \frac{i\omega}{2}[1 + \varepsilon\tilde{y}_n(t)], \quad n = 1, \ldots, N_1 = N - N_0. \tag{31b}$$

Treating ε as an *infinitesimal* constant, it is then a matter of trivial if tedious algebra to get the equations of motion characterizing the evolution of the N_0 coordinates $\tilde{x}_n(t)$ and the N_1 coordinates $\tilde{y}_n(t)$ (where of course $N_0 + N_1 = N$, see (31b)):

$$\ddot{\tilde{x}}_n = \frac{3}{2}[\tilde{\rho} - (N_0 - 2)i\omega]\dot{\tilde{x}}_n - \frac{3}{2}i\omega\left[\tilde{\rho} - \left(N_0 - \frac{4}{3}\right)i\omega\right]\tilde{x}_n + 2\sum_{\ell=1,\ell\neq n}^{N_0}\left[\frac{(\dot{\tilde{x}}_n - i\omega\dot{x}_n)(\dot{\tilde{x}}_\ell - i\omega\dot{x}_\ell)}{\tilde{x}_n - \tilde{x}_\ell}\right], \quad n = 1, \ldots, N_0, \tag{32a}$$

$$\ddot{\tilde{y}}_n = \frac{3}{2}(\tilde{\rho} - N_0 i\omega)\dot{\tilde{y}}_n + 2\sum_{\ell=1,\ell\neq n}^{N_1}\left(\frac{\dot{\tilde{y}}_n\dot{\tilde{y}}_\ell}{\tilde{y}_n - \tilde{y}_\ell}\right), \quad n = 1, \ldots, N_1, \tag{32b}$$

where

$$\tilde{\rho} \equiv \tilde{\rho}(t) = \sum_{j=1}^{N_0}\left[\frac{\dot{\tilde{x}}_j(t)}{\tilde{x}_j(t)}\right]. \tag{32c}$$

This is a *new* (of course *autonomous*, *solvable* and *isochronous*) model of goldfish type. Note that the system (32a) involves only the N_0 coordinates $\tilde{x}_n$, while the system (32b) involves, in addition to the N_1 coordinates $\tilde{y}_n$, also the N_0 coordinates $\tilde{x}_n$, albeit only via the quantity $\tilde{\rho}$, see (32c).

Remark 2.9. It is easily seen — via a treatment analogous to that of Remarks 2.3 and 2.8 — that the system (32a) with (32c) entails the following time evolution of the quantity $\tilde{\rho} \equiv \tilde{\rho}(t)$:

$$2\dot{\tilde{\rho}} = [\tilde{\rho} + (N_0 - 1)i\omega]^2 + \omega^2 \tag{33a}$$

implying

$$\tilde{\rho}(t) = -\omega\left\{(N_0 - 1)i + \operatorname{cotg}\left[\frac{\omega(t + \tilde{t}_0)}{2}\right]\right\}, \tag{33b}$$

where of course the constant $\tilde{t}_0$ is related to the initial data $\tilde{x}_n(0)$, $\dot{\tilde{x}}_n(0)$, $n = 1, \ldots, N_0$, by the formula

$$\operatorname{cotg}\left(\frac{\omega\tilde{t}_0}{2}\right) = -(N_0 - 1)i - \frac{\tilde{\rho}(0)}{\omega} \tag{33c}$$

with (32c). The insertion of this expression, (33b), in (32a) and (32b) transforms these two *autonomous* systems of Newtonian equations of motion (the second of which was coupled to the first via (32c)) into two, apparently decoupled, *nonautonomous* systems, featuring however in their equations of motion the parameter $\tilde{t}_0$ defined in terms of the initial data $\tilde{x}_n(0), \dot{\tilde{x}}_n(0), n = 1, \ldots, N_0$ by the formula (33c) (but this parameter $\tilde{t}_0$ only entails a, generally *complex*, shift of the time coordinate; hence its role is quite marginal). Note that a condition necessary and sufficient to ensure that these equations of motion do not become singular over time is that the initial data yield for the quantity $\tilde{t}_0$ a value which is *not real* (i.e., the right-hand side of (33c) should *not* be *real*). Also note that, if consideration is limited to the *nonautonomous* system of goldfish type (32b) with (33b), reading

$$\ddot{\tilde{y}}_n = -\frac{3}{2}\omega \left\{ i + \text{ cotg}\left[\frac{\omega(t+\tilde{t}_0)}{2}\right]\right\} \dot{\tilde{y}}_n + 2 \sum_{\ell=1,\ell\neq n}^{N_1} \left(\frac{\dot{\tilde{y}}_n \dot{\tilde{y}}_\ell}{\tilde{y}_n - \tilde{y}_\ell}\right), \quad n = 1, \ldots, N_1, \tag{34}$$

the quantity $\tilde{t}_0$ can be evidently considered just an *arbitrary* constant (playing a quite marginal role), as well as the positive integer N_1. This *nonautonomous* many-body problem of goldfish type is of course *solvable* and *isochronous* (to the best of my knowledge, a new finding).

Finally, let us also display the *solvable isochronous* dynamical system satisfied by the N dependent variables $\tilde{c}_m(t)$ related to the N variables $\tilde{z}_n(t)$ (evolving according to the *isochronous* system of goldfish type (23)) by the analog of the relation (16a), reading

$$\tilde{\psi}(z,t) = \prod_{n=1}^{N} [z - \tilde{z}_n(t)] = z^N + \sum_{m=0}^{N} \tilde{c}_m(t) z^{N-m}, \tag{35a}$$

which of course entails

$$\sum_{n=1}^{N} (\tilde{z}_n) = -\tilde{c}_1 \tag{35b}$$

and (see, if need be, [4, Eq. (A.11)])

$$\sum_{n=1}^{N} \left(\frac{\dot{\tilde{z}}_n}{\tilde{z}_n}\right) = \frac{\dot{\tilde{c}}_N}{\tilde{c}_N}. \tag{35c}$$

Again, the derivation from (23a) of the following nonlinear system of Newtonian equations of motion for the N coefficients $\tilde{c}_m$ is a *standard*, if tedious, task, facilitated by the relevant identities displayed in the [4, Appendix A]. This system reads as follows:

$$\begin{aligned}
&\ddot{\tilde{c}}_m - \left[\frac{3\dot{\tilde{c}}_N}{2\tilde{c}_N} + \tilde{c}_1 + \frac{i\omega}{2}(3N-2-4m)\right]\dot{\tilde{c}}_m \\
&\quad + \left\{\frac{3\dot{\tilde{c}}_N}{\tilde{c}_N}\tilde{c}_1 - 2\tilde{c}_1^2 - 2\dot{\tilde{c}}_1 - i\omega\left[\frac{3m\dot{\tilde{c}}_N}{\tilde{c}_N} + (3N-m-2)\tilde{c}_1\right]\omega^2 m\left(\frac{3N-2}{2} - m\right)\right\}\tilde{c}_m \\
&\quad - \left\{3(m+1)\frac{\dot{\tilde{c}}_N}{\tilde{c}_N} + 2(m-1)\tilde{c}_1 + i\omega\left[4m^2 - 3(N-2)m + 5N + 2\right]\right\}\tilde{c}_{m+1} \\
&\quad + 2(2m+1)\dot{\tilde{c}}_{m+1} + 4m(m+2)\tilde{c}_{m+2} = 0, \quad \tilde{c}_{N+1} = \tilde{c}_{N+2} = 0, \quad m = 1, \ldots, N.
\end{aligned} \tag{36}$$

Remark 2.10. An equivalent, but *nonautonomous*, version of these equations of motion obtains via the observation that (23b) and (28) with (35b) and (35c) entail the relation

$$\frac{\dot{\tilde{c}}_N(t)}{\tilde{c}_N(t)} = 2\tilde{c}_1(t) + \omega\left\{(N-1)i - \mathrm{cotg}\left[\frac{\omega(t+\tilde{t}_0)}{2}\right]\right\}, \tag{37a}$$

of course with

$$\omega\mathrm{cotg}\left(\frac{\omega\tilde{t}_0}{2}\right) = (N-1)i\omega - \left[\frac{\dot{\tilde{c}}_N(0)}{\tilde{c}_N(0)} - 2\tilde{c}_1(0)\right]. \tag{37b}$$

3. Proofs

In this section, we provide proofs of the findings reported but not proven in the preceding Sec. 2, i.e., essentially Proposition 2.1 and the findings about the equilibria (Propositions 2.2 and 2.4).

The starting point of our treatment is the trivially solvable matrix ODE

$$\dot{W} = W^2, \tag{38a}$$

the solution of which obviously reads

$$W(t) = W(0)[I - tW(0)]^{-1}. \tag{38b}$$

Here I is the *unit* $(2N)\times(2N)$ matrix, and here and below $W \equiv W(t)$ is a $(2N)\times(2N)$ matrix.

It is now convenient to introduce the following block-matrix representation of the $(2N)\times(2N)$ matrix $W \equiv W(t)$ in terms of two $N\times N$ matrices $U \equiv U(t), V \equiv V(t)$:

$$W = \begin{pmatrix} V & U \\ U & V \end{pmatrix}. \tag{39}$$

This yields the following system of matrix ODEs for the two $N\times N$ matrices $U \equiv U(t), V \equiv V(t)$:

$$\dot{U} = UV + VU, \quad \dot{V} = U^2 + V^2. \tag{40}$$

And it is easily seen (from (38b) and a bit of standard matrix algebra) that the solution of the corresponding initial-value problem yields the following expression of the $N\times N$ matrix $U \equiv U(t)$:

$$U(t) = [I - tV(0)]^{-1}U(0)\cdot\{I - tV(0) - t^2U(0)[I - tV(0)]^{-1}U(0)\}^{-1}. \tag{41}$$

Here and below I is the *unit* $N\times N$ matrix.

We now introduce the N eigenvalues $z_n(t)$ of this $N\times N$ matrix $U(t)$ by setting

$$U(t) = R(t)Z(t)[R(t)]^{-1}, \tag{42a}$$

$$Z(t) = \mathrm{diag}[z_n(t)], \quad Z_{nm}(t) = \delta_{nm}z_n(t). \tag{42b}$$

Remark 3.1. The diagonalizing matrix $R(t)$ is only defined up to right-multiplication by an *arbitrary diagonal* matrix $D \equiv D(t)$, $D_{nm} = \delta_{nm}d_n$, namely it could be replaced by the matrix $\check{R} = RD$ entailing $\check{R}^{-1} = D^{-1}R^{-1}$.

Then, via an analogous formula (featuring the same matrix $R(t)$ which diagonalizes the matrix $U(t)$, see (42a)) we introduce the $N \times N$ (generally nondiagonal) matrix $Y \equiv Y(t)$ by setting

$$V(t) = R(t)Y(t)[R(t)]^{-1}, \tag{43a}$$

$$Y_{nm}(t) = \delta_{nm} y_n(t) + (1 - \delta_{nm}) Y_{nm}(t). \tag{43b}$$

It is moreover convenient to introduce the $N \times N$ matrix $M \equiv M(t)$ by setting

$$M(t) = [R(t)]^{-1} \dot{R}(t), \tag{44a}$$

$$M_{nm}(t) = \delta_{nm} \mu_n(t) + (1 - \delta_{nm}) M_{nm}(t). \tag{44b}$$

Remark 3.2. The freedom entailed by the possibility to right-multiply the matrix R by an *arbitrary diagonal* matrix D (see Remark 3.1) entails that one is free to assign arbitrarily the diagonal elements $\mu_n(t)$ of the matrix $M(t)$, since the change from R to $\check{R}$ (see Remark 3.1) entails the corresponding change from μ_n to $\check{\mu}_n = \mu_n + \dot{d}_n/d_n$.

Via (42a), (43a) and (44a) the matrix ODEs (40) then become

$$\dot{Z} + [M, Z] = ZY + YZ, \tag{45a}$$

$$\dot{Y} + [M, Y] = Z^2 + Y^2. \tag{45b}$$

Let us now look at the diagonal and off-diagonal parts of these two matrix ODEs.

The diagonal part of (45a) yields

$$\dot{z}_n = 2 z_n y_n, \quad n = 1, \ldots, N \tag{46a}$$

entailing

$$y_n = \frac{\dot{z}_n}{2 z_n}, \quad n = 1, \ldots, N, \tag{46b}$$

The off-diagonal part of (45a) yields, componentwise,

$$(z_m - z_n) M_{nm} = (z_n + z_m) Y_{nm}, \quad n \neq m, \quad n, m = 1, \ldots, N, \tag{47a}$$

entailing

$$M_{nm} = -\left(\frac{z_n + z_m}{z_n - z_m}\right) Y_{nm}, \quad n \neq m, \quad n, m = 1, \ldots, N. \tag{47b}$$

The diagonal part of (45b) yields

$$\dot{y}_n = z_n^2 + y_n^2 + \sum_{\ell=1, \ell \neq n}^{N} (Y_{n\ell} Y_{\ell n} - M_{n\ell} Y_{\ell n} + Y_{n\ell} M_{\ell n}), \quad n = 1, \ldots, N, \tag{48a}$$

entailing, via (46b) and (47b),

$$\dot{y}_n = z_n^2 + \frac{\dot{z}_n^2}{4 z_n^2} + \sum_{\ell=1, \ell \neq n}^{N} \left[Y_{n\ell} Y_{\ell n} \left(1 + 2 \frac{z_n + z_\ell}{z_n - z_\ell}\right)\right], \quad n = 1, \ldots, N, \tag{48b}$$

hence, via (46a) and (46b),

$$\ddot{z}_n = \frac{3}{2}\frac{\dot{z}_n^2}{z_n} + 2z_n^3 + 2z_n \sum_{\ell=1,\ell\neq n}^{N} \left[Y_{n\ell}Y_{\ell n}\left(3 + \frac{4z_\ell}{z_n - z_\ell}\right)\right], \quad n = 1, \ldots, N. \tag{49}$$

Finally, the off-diagonal part of (45b) yields, componentwise,

$$\dot{Y}_{nm} = \sum_{k=1}^{N} (Y_{nk}Y_{km} - M_{nk}Y_{km} + Y_{nk}M_{km}), \quad n \neq m, \quad n, m = 1, \ldots, N, \tag{50a}$$

hence, via (43b), (44b), (46b), (47b) and a bit of algebra,

$$\frac{\dot{Y}_{nm}}{Y_{nm}} = \frac{\dot{z}_n}{z_n} + \frac{\dot{z}_m}{z_m} - \frac{\dot{z}_n - \dot{z}_m}{z_n - z_m} - \mu_n + \mu_m + \sum_{\ell=1,\ell\neq n,m}^{N} \left[\frac{Y_{n\ell}Y_{\ell m}}{Y_{nm}}\left(3 + \frac{2z_\ell}{z_n - z_\ell} + \frac{2z_\ell}{z_m - z_\ell}\right)\right], \quad n \neq m, \quad n, m = 1, \ldots, N. \tag{50b}$$

We now note that the system of N ODEs (49) looks like a set of Newtonian equations of motion for the coordinates z_n, with the quantities $Y_{n\ell}Y_{\ell n}$ playing the role of "coupling constants". But the quantities $Y_{n\ell}$ are *not* constants; indeed their time evolution is determined by the system of ODEs (50b). To get an autonomous system of Newtonian evolution equations involving only the N coordinates z_n one must therefore find — if at all possible — a way to get rid of the $N(N-1)$ quantities Y_{nm}, namely a way to express them in terms of the coordinates z_n and their time derivatives $\dot{z}_n$ (while of course satisfying the $N(N-1)$ ODEs (50b), using, if need be, the freedom to assign the N quantities μ_n, see Remark 3.2). Somewhat miraculously the following *ansatz* allows to realize this goal:

$$\mu_n = 0, \quad n = 1, \ldots, N, \tag{51a}$$

$$Y_{nm} = \frac{1}{2}\left(\frac{\dot{z}_n}{z_n} + 2z_n\right)^{1/2}\left(\frac{\dot{z}_m}{z_m} + 2z_m\right)^{1/2}, \quad n \neq m, \quad n, m = 1, \ldots, N. \tag{51b}$$

Indeed via this *ansatz* the system (49) coincides with the equations of motion (2a) while, via this very expression, (2a), of $\ddot{z}_n$, it is rather easily seen that the *ansatz* (51) satisfies identically the $N(N-1)$ equations (50b). This concludes the main part of our proof of Proposition 2.1.

There remains to express the two $N \times N$ matrices $U(0)$ and $V(0)$ featured by the solution (41) in terms of the initial data $z_n(0), \dot{z}_n(0), n = 1, \ldots, N$. The simplest (and clearly permitted) way is to assign $R(0) = I$, thereby entailing, via (42a), $U(0) = Z(0), V(0) = Y(0)$. Then the solution (41) reproduces the formula (3a), while clearly (42b) yields the expression (3b) of $Z(0)$ and, via (43b), (46b) and (51b), one gets the expression (3c) of $Y(0)$.

The proof of Proposition 2.1 is thereby completed.

Next, let us consider the equilibria. Let us re-emphasize that the equilibrium coordinates $z_n = \bar{z}_n, \dot{z}_n = 0$ are always defined up to permutations.

For the *nonisochronous* case of Sec. 2.1, we begin by noting that the equilibrium configuration $z_n = \bar{z}_n, \dot{z}_n = 0$ corresponding to the system (2b) is clearly characterized by the following system of N algebraic equations for the N (time-independent) coordinates $\bar{z}_n$:

$$\bar{z}_n^2 \left\{ -2\bar{z}_n + 3\sum_{k=1}^{N}(\bar{z}_k) + 4\sum_{\ell=1,\ell\neq n}^{N}\left(\frac{\bar{z}_\ell^2}{\bar{z}_n - \bar{z}_\ell}\right)\right\} = 0, \quad n = 1,\ldots,N. \tag{52a}$$

Assume now that K of the coordinates $\bar{z}_n$ do *not* vanish, while the remaining ones do vanish, say, for definiteness, $\bar{z}_n = 0$ for $n = K+1,\ldots,N$ (with $0 \leq K \leq N$). The $K = 0$ case corresponds to the special case of Proposition 2.2 with $\bar{z} = 0$. Hence hereafter $K \geq 1$. Then the above set of equations is reduced to the following system of K algebraic equations for the K numbers $\bar{z}_k$:

$$-2\bar{z}_k + 3\sum_{j=1}^{K}(\bar{z}_j) + 4\sum_{\ell=1,\ell\neq k}^{K}\left(\frac{\bar{z}_\ell^2}{\bar{z}_k - \bar{z}_\ell}\right) = 0, \quad k = 1,\ldots,K, \tag{53a}$$

or equivalently

$$-2(2K-3)\bar{z}_k - \sum_{j=1}^{K}(\bar{z}_j) + 4\sum_{\ell=1,\ell\neq k}^{K}\left(\frac{\bar{z}_k^2}{\bar{z}_k - \bar{z}_\ell}\right) = 0, \quad k = 1,\ldots,K. \tag{53b}$$

The equivalence is of course implied by the identity

$$\bar{z}_\ell^2 = \bar{z}_k^2 - (\bar{z}_k - \bar{z}_\ell)(\bar{z}_k + \bar{z}_\ell). \tag{54}$$

On the other hand by summing the K equations of the system (53a) and using the identity

$$2\sum_{k,\ell=1,\ell\neq k}^{K}\left(\frac{\bar{z}_\ell^2}{\bar{z}_k - \bar{z}_\ell}\right) = -(K-1)\sum_{k=1}^{K}(\bar{z}_k) \tag{55}$$

(clearly implied by (54)), we get $K\sum_{j=1}^{K}(\bar{z}_j) = 0$ namely $\sum_{j=1}^{K}(\bar{z}_j) = 0$. Hence (53b) can now be simplified to read

$$4\sum_{\ell=1,\ell\neq k}^{K}\left(\frac{\bar{z}_k}{\bar{z}_k - \bar{z}_\ell}\right) = 2(2K-3), \quad k = 1,\ldots,K. \tag{56}$$

Finally summing these K equations and using the identity

$$2\sum_{k,\ell=1,\ell\neq k}^{K}\left(\frac{\bar{z}_k}{\bar{z}_k - \bar{z}_\ell}\right) = K(K-1) \tag{57a}$$

we clearly get the relation

$$2K(K-1) = 2K(2K-3) \quad \text{hence} \quad K = 0 \text{ or } K = 2. \tag{57b}$$

It is then trivial to complete the proof of Proposition 2.2 by noting that (56) with $K = 2$ implies $\bar{z}_1 = \bar{z}, \bar{z}_2 = -\bar{z}$ with $\bar{z}$ arbitrary.

In the *isochronous* case of Sec. 2.2 it is convenient to utilize the rescaling (29) of the coordinates $\bar{z}_n$ identifying the equilibrium configurations. Then clearly from (23a) we get the following version of the system of N algebraic equations determining the N numbers ζ_n:

$$\zeta_n(1-\zeta_n)\left\{3N-4-3\sigma+2\zeta_n+4\sum_{\ell=1,\ell\neq n}^{N}\left[\frac{\zeta_\ell(1-\zeta_\ell)}{\zeta_n-\zeta_\ell}\right]\right\}=0,\quad n=1,\ldots,N, \tag{58a}$$

where we set

$$\sigma\equiv\sum_{n=1}^{N}(\zeta_n). \tag{58b}$$

Our task here is to identify all the solutions of this system of N algebraic equations. Each solution is characterized by N values ζ_n; of course they are always defined up to an arbitrary permutation of the N indices n. Clearly there are two values of the coordinates ζ_n which play a special role, namely the values 0 and 1. Therefore, hereafter we assume that N_0 respectively N_1 of the N numbers ζ_n have these special values 0 respectively 1, say, for definiteness,

$$\zeta_n=0\quad\text{for } N-N_0+1\leq n\leq N, \tag{59a}$$

$$\zeta_n=1\quad\text{for } N-N_0-N_1+1\leq n\leq N-N_0, \tag{59b}$$

where of course N_0 and N_1 are two *non-negative* integers the sum of which does not exceed N so that

$$K=N-N_0-N_1,\quad 1\leq K\leq N. \tag{59c}$$

Implicit in this *ansatz* is the assumption that for $\zeta_n=0$ or $\zeta_n=1$ the nth equation of the algebraic systems (58a) is automatically satisfied; moreover note that, by assuming here that K is *positive*, we are hereafter excluding from consideration the *trivial* case in which *all* the N numbers ζ_n either *vanish* or equal *unity* (i.e., the solution of *first type* in Proposition 2.4). Hence our task is now reduced to finding the K solutions of the following system of K algebraic equations

$$3(K+N_0)-4-3S+2\zeta_k+4\sum_{\ell=1,\ell\neq k}^{K}\left[\frac{\zeta_\ell(1-\zeta_\ell)}{\zeta_k-\zeta_\ell}\right]=0,\quad k=1,\ldots,K, \tag{60a}$$

with

$$S\equiv\sum_{k=1}^{K}(\zeta_k). \tag{60b}$$

Note that the number N_1 has disappeared from these equations (because $\sigma=S+N_1$, see (58b), (60b) and (59)), and that our treatment entails now the requirement that the solution of this system feature no number ζ_k which vanishes or equals unity,

$$\zeta_k\neq 0,\quad \zeta_k\neq 1,\quad k=1,\ldots,K. \tag{60c}$$

For $K=1$ the system (60) reduces to a single (linear) equation and it clearly yields $\zeta_1=3\ N_0-1$, as indicated above. This is the solution of *second type* in Proposition 2.4.

Likewise, for $K = 2$ the system (60) can be easily solved, and it is thereby seen to imply $N_0 = 0$ and to yield for ζ_1 and ζ_2 the two values (30) with ζ an arbitrary number. This is the solution of *third type* in Proposition 2.4.

To complete the proof of Proposition 2.4 there remains to show that the system (60) has no solution for $K \geq 3$. We proceed *per absurdum*, introducing the (monic) polynomial of degree K featuring the K numbers ζ_k determined by the system (60) as its K zeros:

$$\psi(\zeta) = \prod_{k=1}^{K}(\zeta - \zeta_k) = \sum_{k=0}^{K}(c_k \zeta^{K-k}), \quad c_0 = 1, \tag{61a}$$

and then showing that the assumption that such a polynomial exist leads to contradictions. Note that this definition, together with (60b), entails the relation

$$c_1 = -S. \tag{61b}$$

Next — using (61b) and, if need be, the formulas (A.4), (A.6a), (A.8b) and (A.8f) of [4] — we note that to the algebraic system (60a) satisfied by the K numbers ζ_k there corresponds for the polynomial $\psi(\zeta)$ the following second-order ODE:

$$\zeta\psi'' + \left(\frac{3N_0 - K - c_1}{2}\right)\psi' \tag{62a}$$

$$= \zeta^2\psi'' - (2K-3)\zeta\psi' + K(K-2)\psi. \tag{62b}$$

Here of course the appended primes denote differentiations with respect to the argument ζ of the polynomial $\psi(\zeta)$.

Likewise — using, if need be, the formulas (A.13g), (A.13c), (A.13h), (A.13d) and (A.13a) of [4] — it is easily seen that this ODE yields for the coefficients c_k, see (61a), the following recursion relation:

$$(K+1-k)\left(\frac{3N_0 + K - c_1}{2} - k\right)c_{k-1} = k(k-2)c_k, \quad k = 1, 2, \ldots. \tag{63}$$

For $k = 1$ this formula yields (using $c_0 = 1$, see (61a))

$$c_1 = K\left(1 + \frac{3N_0}{K-2}\right). \tag{64}$$

Then, for $k = 2$, we get from (63) either $c_1 = 0$ which is inconsistent with (64) for $K \geq 3$ and $N_0 \geq 0$, or

$$c_1 = 3N_0 + K - 4 \tag{65a}$$

namely, via (64),

$$2K + 3N_0 = 4 \tag{65b}$$

which also cannot be satisfied for $K \geq 3$ and $N_0 \geq 0$. The proof of Proposition 2.4 is thereby completed.

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