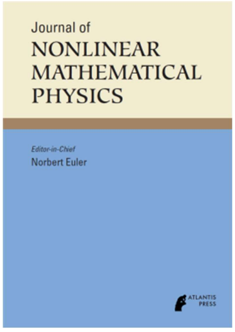

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Arthemy V. Kiselev, Andrey O. Krutov

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Non-Abelian Lie algebroids over jet spaces

Arthemy V. Kiselev
Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen
P.O.Box 407, Groningen, 9700 AK, The Netherlands
A.V.Kiselev@rug.nl

Andrey O. Krutov
Department of Higher Mathematics, Ivanovo State Power University
Rabfakovskaya str. 34, Ivanovo, 153003, Russia
krutov@math.ispu.ru

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We associate Hamiltonian homological evolutionary vector fields – which are the non-Abelian variational Lie algebroids' differentials – with Lie algebra-valued zero-curvature representations for partial differential equations.

Keywords: Zero-curvature representation, gauge transformation, Lie algebroid, homological vector field, master equation

2000 Mathematics Subject Classification: 37K10, 81T70, also 53D17, 58A20, 70S15, 81T13.

Introduction

Lie algebra-valued zero-curvature representations for partial differential equations (PDE) are the input data for solving Cauchy's problems by the inverse scattering method [42]. For a system of PDE with unknowns in two independent variables to be kinematically integrable, a zero-curvature representation at hand must depend on a spectral parameter which is non-removable under gauge transformations. In the paper [33] M. Marvan developed a remarkable method for inspection whether a parameter in a given zero-curvature representation α is (non)removable; this technique refers to a cohomology theory generated by a differential ∂_α, which was explicitly constructed for every α.

In this paper we show that zero-curvature representations for PDE give rise to a natural class of non-Abelian variational Lie algebroids. In section 1 (see Fig. 1 on p. 6) we list all the components of such structures (cf. [25]); in particular, we show that Marvan's operator ∂_α is the anchor. In section 2, non-Abelian variational Lie algebroids are realized via BRST-like homological evolutionary vector fields Q on superbundles à la [5]. Having enlarged the BRST-type setup to a geometry which goes in a complete parallel with the standard BV-zoo ([4], see also [2]), in section 3 we extend the vector field Q to the evolutionary derivation $\widehat{Q}(\cdot) \cong [\![\widehat{S}, \cdot]\!]$ whose Hamiltonian functional $\widehat{S}$ satisfies the classical master-equation $[\![\widehat{S},\widehat{S}]\!] = 0$. We then address that equation's gauge symmetry invariance and $\widehat{Q}$-cohomology automorphisms ([29], cf. [13] and [18]), which yields the next generation of Lie algebroids, see Fig. 2 on p. 15.

Two appendices follow the main exposition. We first recall the notion of Lie algebroids over usual smooth manifolds. (Appendix A.1 concludes with an elementary explanation why the classical construction stops working over infinite jet spaces or over PDE such as gauge systems.) Secondly, we describe the idea of parity-odd neighbours to vector spaces and their use in $\mathbb{Z}_2$-graded superbundles [39]. In particular, we recall how Lie algebroids or Lie algebroid differentials are realised in terms of homological vector fields on the total spaces of such superbundles [37].

In the earlier work [25] by the first author and J. W. van de Leur, classical notions, operations, and reasonings which are contained in both appendices were upgraded from ordinary manifolds to jet bundles, which are endowed with their own, restrictive geometric structures such as the Cartan connection $\nabla_{\mathscr{C}}$ and which harbour systems of PDE. We prove now that the geometry of Lie algebra-valued connection $\mathfrak{g}$-forms α satisfying zero-curvature equation (1.1) gives rise to the geometry of solutions $\widehat{S}$ for the classical master-equation

$$\mathscr{E}_{\text{CME}} = \bigl\{ i\hbar\Delta\widehat{S}\big|_{\hbar=0} = \tfrac{1}{2}[\![\widehat{S},\widehat{S}]\!] \bigr\}, \tag{0.1}$$

see Theorem 3.1 on p. 11 below. It is readily seen that realization (0.1) of the gauge-invariant setup is the classical limit of the full quantum picture as $\hbar \to 0$; the objective of quantization $\widehat{S} \longmapsto S^{\hbar}$ is a solution of the quantum master-equation

$$\mathscr{E}_{\text{QME}} = \bigl\{ i\hbar\Delta S^{\hbar} = \tfrac{1}{2}[\![S^{\hbar},S^{\hbar}]\!] \bigr\} \tag{0.2}$$

for the true action functional $S^{\hbar}$ at $\hbar \neq 0$. Its construction involves quantum, noncommutative objects such as the deformations $\mathfrak{g}_{\hbar}$ of Lie algebras together with deformations of their duals (cf. [10]). (In fact, we express the notion of non-Abelian variational Lie algebroids in terms of the homological evolutionary vector field $\widehat{Q}$ and classical master-equation (0.1) viewing this construction as an intermediate step towards quantization.) A transition from the semiclassical to quantum picture results in $\mathfrak{g}_{\hbar}$-valued connections, quantum gauge groups, quantum vector spaces for values of the wave functions in auxiliary linear problems (1.2), and quantum extensions of physical fields.[a]

1. Preliminaries

Let us first briefly recall some definitions (see [6,19,34] and [33] for detail); this material is standard so that we now fix the notation.

1.1. *The geometry of infinite jet space $J^{\infty}(\pi)$*

Let M^n be a smooth real n-dimensional orientable manifold. Consider a smooth vector bundle $\pi\colon E^{n+m} \to M^n$ with m-dimensional fibres and construct the space $J^{\infty}(\pi)$ of infinite jets of sections for π. A convenient organization of local coordinates is as follows: let x^i be some coordinate system on a chart in the base M^n and denote by u^j the coordinates along a fibre of the bundle π so that the variables u^j play the rôle of unknowns; one obtains the collection u^j_{σ} of jet variables

[a]Lie algebra-valued connection one-forms are the main objects in classical gauge field theories. Such physical models are called *Abelian* – e.g., Maxwell's electrodynamics – or *non-Abelian* – here, consider the Yang–Mills theories with structure Lie groups $SU(2)$ or $SU(3)$ – according to the commutation table for the underlying Lie algebra. This is why we say that variational Lie algebroids are (*non-*)*Abelian*— referring to the Lie algebra-valued connection one-forms α in the geometry of gauge-invariant zero-curvature representations for PDE.

along fibres of the vector bundle $J^\infty(\pi) \to M^n$ (here $|\sigma| \geqslant 0$ and $u^j_\varnothing \equiv u^j$). In this setup, the *total derivatives* D_{x^i} are commuting vector fields $D_{x^i} = \nabla_{\mathscr{C}}(\partial/\partial x^i) = \partial/\partial x^i + \sum_{j,\sigma} u^j_{\sigma i}\,\partial/\partial u^j_\sigma$ on $J^\infty(\pi)$.

Consider a system of partial differential equations

$$\mathscr{E} = \Big\{ F^\ell(x^i, u^j, \ldots, u^j_\sigma, \ldots) = 0, \quad \ell = 1, \ldots, r < \infty \Big\};$$

without any loss of generality for applications we assume that the system at hand satisfies mild assumptions which are outlined in [19,34]. Then the system $\mathscr{E}$ and all its differential consequences $D_\sigma(F^\ell) = 0$ (thus presumed existing, regular, and not leading to any contradiction in the course of derivation) generate the infinite prolongation $\mathscr{E}^\infty$ of the system $\mathscr{E}$.

Let us denote by $\bar{D}_{x^i}$ the restrictions of total derivatives D_{x^i} to $\mathscr{E}^\infty \subseteq J^\infty(\pi)$. We recall that the vector fields $\bar{D}_{x^i}$ span the Cartan distribution $\mathscr{C}$ in the tangent space $T\mathscr{E}^\infty$. At every point $\theta^\infty \in \mathscr{E}^\infty$ the tangent space $T_{\theta^\infty}\mathscr{E}^\infty$ splits in a direct sum of two subspaces. The one which is spanned by the Cartan distribution $\mathscr{E}^\infty$ is *horizontal* and the other is *vertical*: $T_{\theta^\infty}\mathscr{E}^\infty = \mathscr{C}_{\theta^\infty} \oplus V_{\theta^\infty}\mathscr{E}^\infty$. We denote by $\Lambda^{1,0}(\mathscr{E}^\infty) = \operatorname{Ann}\mathscr{C}$ and $\Lambda^{0,1}(\mathscr{E}^\infty) = \operatorname{Ann} V\mathscr{E}^\infty$ the $C^\infty(\mathscr{E}^\infty)$-modules of contact and horizontal one-forms which vanish on $\mathscr{C}$ and $V\mathscr{E}^\infty$, respectively. Denote further by $\Lambda^r(\mathscr{E}^\infty)$ the $C^\infty(\mathscr{E}^\infty)$-module of r-forms on $\mathscr{E}^\infty$. There is a natural decomposition $\Lambda^r(\mathscr{E}^\infty) = \bigoplus_{q+p=r} \Lambda^{p,q}(\mathscr{E}^\infty)$, where $\Lambda^{p,q}(\mathscr{E}^\infty) = \bigwedge^p \Lambda^{1,0}(\mathscr{E}^\infty) \wedge \bigwedge^q \Lambda^{0,1}(\mathscr{E}^\infty)$. This implies that the de Rham differential $\bar{\mathrm{d}}$ on $\mathscr{E}^\infty$ is subjected to the decomposition $\bar{\mathrm{d}} = \bar{\mathrm{d}}_h + \bar{\mathrm{d}}_{\mathscr{C}}$, where $\bar{\mathrm{d}}_h \colon \Lambda^{p,q}(\mathscr{E}^\infty) \to \Lambda^{p,q+1}(\mathscr{E}^\infty)$ is the horizontal differential and $\bar{\mathrm{d}}_{\mathscr{C}} \colon \Lambda^{p,q}(\mathscr{E}^\infty) \to \Lambda^{p+1,q}(\mathscr{E}^\infty)$ is the vertical differential. In local coordinates, the differential $\bar{\mathrm{d}}_h$ acts by the rule

$$\bar{\mathrm{d}}_h = \sum\nolimits_i \mathrm{d}x^i \wedge \bar{D}_{x^i}.$$

We shall use this formula in what follows. By definition, we put $\bar{\Lambda}(\mathscr{E}^\infty) = \bigoplus_{q\geqslant 0} \Lambda^{0,q}(\mathscr{E}^\infty)$ and we denote by $\overline{H}^n(\cdot)$ the senior d_h-cohomology groups (also called senior *horizontal cohomology*) for the infinite jet bundles which are indicated in parentheses, cf. [20].

Remark 1.1. The geometry which we analyse in this paper is produced and arranged by using the pull-backs $f^*(\rho)$ of fibre bundles ρ under some mappings f. Typically, the fibres of ρ are Lie algebra-valued horizontal differential forms coming from $\Lambda^*(M^n)$, or similar objects[b] ; in turn, the mappings f are projections to the base M^n of some infinite jet bundles. We employ the standard notion of *horizontal infinite jet bundles* such as $\overline{J^\infty_\xi}(\chi)$ or $\overline{J^\infty_\chi}(\xi)$ over infinite jet bundles $J^\infty(\xi)$ and $J^\infty(\chi)$, respectively; these spaces are present in Fig. 1 on p. 6 and they occur in (the proof of) Theorems 2.1 and 3.1 below. A proof of the convenient isomorphism $\overline{J^\infty_\xi}(\chi) \cong J^\infty(\xi \times_{M^n} \chi) = J^\infty(\xi) \times_{M^n} J^\infty(\chi)$ is written in [27], see also references therein. However, we recall further that, strictly speaking, the entire picture – with fibres which are inhabited by form-valued parity-even or parity-odd (duals of the) Lie algebra $\mathfrak{g}$ – itself is the image of a pull-back under the projection $\pi_\infty \colon J^\infty(\pi) \to M^n$ in the infinite jet bundle over the bundle π of physical fields. In other words, *sections* of those induced bundles are elements of Lie algebra etc., but all coefficients are differential functions in configurations of physical fields (which is obvious, e. g., from (1.1) in Definition 1.1 on the next page). Fortunately, it is the composite geometry of a fibre but not its location over the composite-structure base manifold which plays the main rôle in proofs of Theorems 2.1 and 3.1.

[b]Let us specify at once that the geometries of prototype fibres in the bundles under study are described by $\mathfrak{g}$-, $\mathfrak{g}^*$-, $\Pi\mathfrak{g}$-, or $\Pi\mathfrak{g}^*$-valued (-1)-, zero-, one-, two-, and three-forms ; the degree -1 corresponds to the module $D_1(M^n)$ of vector fields.

It is clear now that an attempt to indicate not only the bundles ξ or χ, $\Pi\chi^*$, $\Pi\xi$, and ξ^* which determine the intrinsic properties of objects but also to display the bundles that generate the pull-backs would make all proofs sound like the well-known poem about the house that Jack built.

Therefore, we *denote* the objects such as p_i or α and their mappings (see p. 9 or p. 12) *as if* they were just sections, $p_i \in \Gamma(\xi)$ and $\alpha \in \Gamma(\chi)$, of the bundles ξ and χ over the base M^n, leaving obvious technical details to the reader.

1.2. *Zero-curvature representations*

Let $\mathfrak{g}$ be a finite-dimensional (complex) Lie algebra. Consider its tensor product (over $\mathbb{R}$) with the exterior algebra of horizontal differential forms $\bar{\Lambda}(\mathscr{E}^\infty)$ on the infinite prolongation of $\mathscr{E}$. This product is endowed with a $\mathbb{Z}$-graded Lie algebra structure by the bracket $[A\mu, B\nu] = [A,B]\,\mu\wedge\nu$, where $\mu,\nu \in \bar{\Lambda}(\mathscr{E}^\infty)$ and $A,B \in \mathfrak{g}$.

Let us focus on the case of $\mathfrak{g}$-valued one-forms. In the tensor product, the Jacobi identity for α, β, $\gamma \in \mathfrak{g}\otimes\Lambda^{0,1}(\mathscr{E}^\infty)$ looks as follows. Let $\alpha = A\mu$, $\beta = B\nu$, $\gamma = C\omega$. We obtain that

$$\begin{aligned}
&[\alpha,[\beta,\gamma]] + [\gamma,[\alpha,\beta]] + [\beta,[\gamma,\alpha]] \\
&\qquad = [A\mu,[B,C]\,\nu\wedge\omega] + [C\omega,[A,B]\,\mu\wedge\nu] + [B\nu,[C,A]\,\omega\wedge\mu] \\
&\qquad = [A,[B,C]]\,\mu\wedge\nu\wedge\omega + [C,[A,B]]\,\omega\wedge\mu\wedge\nu + [B,[C,A]\,\nu\wedge\omega\wedge\mu.
\end{aligned}$$

For the one-forms μ, ν, and γ we have that $\mu\wedge\nu\wedge\gamma = \gamma\wedge\mu\wedge\nu = \nu\wedge\gamma\wedge\mu$ so that the above equality continues with

$$= \big([A,[B,C]] + [C,[A,B]] + [B,[C,A]]\big)\,\mu\wedge\nu\wedge\omega = 0.$$

Indeed, this expression vanishes due to the Jacobi identity of the Lie algebra $\mathfrak{g}$, namely, $[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$.

The horizontal differential d_h acts on elements of $A\otimes\mu \in \mathfrak{g}\otimes\Lambda(\mathscr{E}^\infty)$ as follows:

$$\mathrm{d}_h(A\otimes\mu) = A\otimes\mathrm{d}_h\mu.$$

Definition 1.1. A horizontal one-form $\alpha \in \mathfrak{g}\otimes\Lambda^{0,1}(\mathscr{E}^\infty)$ is called a $\mathfrak{g}$-valued zero-curvature representation for $\mathscr{E}$ if α satisfies the Maurer–Cartan equation

$$\mathscr{E}_{\mathrm{MC}} = \big\{\bar{\mathrm{d}}_h\alpha - \tfrac{1}{2}[\alpha,\alpha] \doteq 0\big\} \tag{1.1}$$

by virtue of equation $\mathscr{E}$ and its differential consequences.

Given a zero-curvature representation $\alpha = A_i\,\mathrm{d}x^i$, the Maurer–Cartan equation $\mathscr{E}_{\mathrm{MC}}$ can be interpreted as the compatibility condition for the linear system

$$\Psi_{x^i} = A_i\Psi, \tag{1.2}$$

where $A_i \in \mathfrak{g}\otimes C^\infty(\mathscr{E}^\infty)$ and Ψ is the wave function, that is, Ψ is a (local) section of the principal fibre bundle $P(\mathscr{E}^\infty, G)$ with action of the gauge Lie group G on fibres; the Lie algebra of G is $\mathfrak{g}$. Then the system of equations

$$D_{x^i}A_j - D_{x^j}A_i + [A_i,A_j] = 0, \qquad 1 \leqslant i < j \leqslant n,$$

is equivalent to Maurer–Cartan's equation (1.1).

1.3. *Gauge transformations*

Let $\mathfrak{g}$ be the Lie algebra of the Lie group G and α be a $\mathfrak{g}$-valued zero-curvature representation for a given PDE system $\mathscr{E}$. A gauge transformation $\Psi \mapsto g\Psi$ of the wave function by an element $g \in C^\infty(\mathscr{E}^\infty, G)$ induces the change

$$\alpha \mapsto \alpha^g = g\cdot\alpha\cdot g^{-1} + \bar{\mathrm{d}}_h g\cdot g^{-1}.$$

The zero-curvature representation α^g is called *gauge equivalent* to the initially given α; the G-valued function g on $\mathscr{E}^\infty$ determines the *gauge transformation* of α. For convenience, we make no distinction between the gauge transformations $\alpha \mapsto \alpha^g$ and G-valued functions g which generate them.

It is readily seen that a composition of two gauge transformations, by using g_1 first and then by g_2, itself is a gauge transformation generated by the G-valued function $g_2\circ g_1$. Indeed, we have that

$$\begin{aligned}(\alpha^{g_1})^{g_2} = (\bar{\mathrm{d}}_h g_1\cdot g_1^{-1} + g_1\cdot\alpha\cdot g_1^{-1})^{g_2} &= \bar{\mathrm{d}}_h g_2\cdot g_2^{-1} + g_2\cdot(\bar{\mathrm{d}}_h g_1\cdot g_1^{-1} + g_1\cdot\alpha\cdot g_1^{-1})\cdot g_2^{-1}\\ &= (\bar{\mathrm{d}}_h g_2\cdot g_1 + g_2\cdot\bar{\mathrm{d}}_h g_1)\cdot g_1^{-1}\cdot g_2^{-1} + g_2\cdot g_1\cdot\alpha\cdot g_1^{-1}\cdot g_2^{-1}\\ &= \bar{\mathrm{d}}_h(g_2\cdot g_1)\cdot(g_2\cdot g_1)^{-1} + (g_2\cdot g_1)\cdot\alpha\cdot(g_2\cdot g_1)^{-1}.\end{aligned}$$

We now consider *infinitesimal* gauge transformations generated by elements of the Lie group G which are close to its unit element 1. Suppose that $g_1 = \exp(\lambda p_1) = 1 + \lambda p_1 + \frac{1}{2}\lambda^2 p_1^2 + o(\lambda^2)$ and $g_2 = \exp(\mu p_2) = 1 + \mu p_2 + \frac{1}{2}\mu^2 p_2^2 + o(\mu^2)$ for some $p_1, p_2 \in \mathfrak{g}$ and $\mu, \lambda \in \mathbb{R}$. The following lemma, an elementary proof of which refers to the definition of Lie algebra, is the key to a construction of the anchors in non-Abelian variational Lie algebroids.

Lemma 1.1. *Let α be a $\mathfrak{g}$-valued zero-curvature representation for a system $\mathscr{E}$. Then the commutant $g_1\circ g_2\circ g_1^{-1}\circ g_2^{-1}$ of infinitesimal gauge transformations g_1 and g_2 is an infinitesimal gauge transformation again.*

Proof. By definition, put $g = g_1\circ g_2\circ g_1^{-1}\circ g_2^{-1}$. Taking into account that $g_1^{-1} = 1 - \lambda p_1 + \frac{1}{2}\lambda^2 p_1^2 + o(\lambda^2)$ and $g_2^{-1} = 1 - \mu p_2 + \frac{1}{2}\mu^2 p_2^2 + o(\mu^2)$, we obtain that

$$g = g_1 g_2 g_1^{-1} g_2^{-2} = 1 + \lambda\mu\cdot(p_1 p_2 - p_2 p_1) + o(\lambda^2 + \mu^2).$$

We finally recall that $[p_1, p_2] \in \mathfrak{g}$, whence follows the assertion. □

An infinitesimal gauge transformation $g = 1 + \lambda p + o(\lambda)$ acts on a given $\mathfrak{g}$-valued zero-curvature representation α for an equation $\mathscr{E}^\infty$ by the formula

$$\begin{aligned}\alpha^g &= \bar{\mathrm{d}}_h(1 + \lambda p + o(\lambda))\cdot(1 - \lambda p + o(\lambda)) + (1 + \lambda p + o(\lambda))\cdot\alpha\cdot(1 - \lambda p + o(\lambda))\\ &= \lambda\bar{\mathrm{d}}_h p + \alpha + \lambda(p\alpha - \alpha p) + o(\lambda) = \alpha + \lambda(\bar{\mathrm{d}}_h p + [p, \alpha]) + o(\lambda).\end{aligned}$$

From the coefficient of λ we obtain the operator $\bar{\partial}_\alpha = \bar{\mathrm{d}}_h + [\cdot, \alpha]$. Lemma 1.1 implies that the image of this operator is closed under commutation in $\mathfrak{g}$, that is, $[\operatorname{im}\bar{\partial}_\alpha, \operatorname{im}\bar{\partial}_\alpha] \subseteq \operatorname{im}\bar{\partial}_\alpha$. Such operators and their properties were studied in [25, 26]. We now claim that the operator $\bar{\partial}_\alpha$ yields the anchor in a non-Abelian variational Lie algebroid, see Fig. 1;

this construction is elementary (see Remark 1.1 on p. 3). Namely, the non-Abelian Lie algebroid $(\pi_\infty^*\circ\chi_\infty^*(\xi), \partial_\alpha, [\,,\,]_{\mathfrak{g}})$ consists of

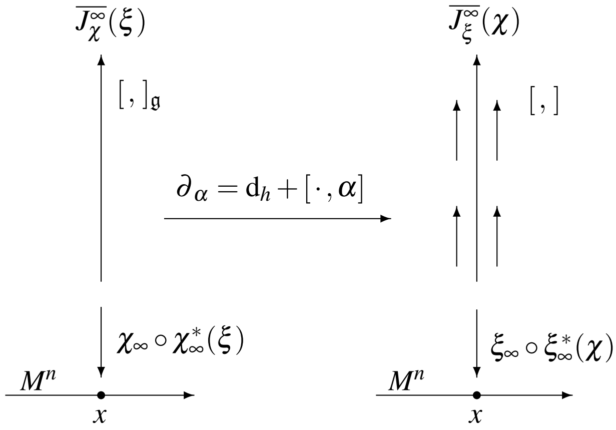

Fig. 1. Non-Abelian variational Lie algebroid.

- the pull-back of the bundle ξ for $\mathfrak{g}$-valued gauge parameters p; the pull-back is obtained by using the bundle χ for $\mathfrak{g}$-forms α and (again by using the infinite jet bundle π_∞ over) the bundle π of physical fields,
- the (restriction $\bar{\partial}_\alpha$ to $\mathscr{E}^\infty \subseteq J^\infty(\pi)$ of the) anchor ∂_α that generates infinitesimal gauge transformations $\dot{\alpha} = \partial_\alpha(p)$ in the bundle χ of $\mathfrak{g}$-valued connection one-forms, and
- the Lie algebra structure $[\,,\,]_\mathfrak{g}$ on the anchor's domain of definition.

We refer to Appendix A.1 for more detail and to p. 18 for discussion on that object's structural complexity.

1.4. *Noether identities for the Maurer–Cartan equation*

In the meantime, let us discuss Noether identities [6, 19, 34] for Maurer–Cartan equation (1.1). Depending on the dimension n of the base manifold M^n, we consider the cases $n = 2$, $n = 3$, and $n > 3$. We suppose that the Lie algebra $\mathfrak{g}$ is equipped[c] with a nondegenerate ad-invariant metric t_{ij}. The paring $\langle\,,\,\rangle$ is defined for elements of $\mathfrak{g}\otimes\Lambda(M^n)$ as follows,

$$\langle A\mu, B\nu\rangle = \langle A,B\rangle\,\mu\wedge\nu,$$

where the coupling $\langle A,B\rangle$ is given by the metric t_{ij} for $\mathfrak{g}$. From the ad-invariance $\langle [A,B],C\rangle = \langle A,[B,C]\rangle$ of the metric t_{ij} we deduce that

$$\begin{aligned}\langle [A\mu,B\nu],C\rho\rangle = \langle [A,B]\,\mu\wedge\nu, C\rho\rangle = \langle [A,B],C\rangle\,\mu\wedge\nu\wedge\rho = \langle A,[B,C]\rangle\,\mu\wedge\nu\wedge\rho \\ = \langle A\mu,[B,C]\,\nu\wedge\rho\rangle = \langle A\mu,[B\nu,C\rho]\rangle.\end{aligned}$$

Let us denote by $\mathscr{F} = -\mathrm{d}_h\alpha + \frac{1}{2}[\alpha,\alpha]$ the left-hand side of Maurer–Cartan equation (1.1). We recall from section 1.3 that $\dot{\alpha} = \partial_\alpha(p)$ is a gauge symmetry of Maurer–Cartan equation (1.1). Moreover, for all $n > 1$ the operator $\partial_\alpha^\dagger$ produces a Noether identity for (1.1), which is readily seen from the following statement.

[c]Notice that the Lie algebra $\mathfrak{g}$ is canonically identified with its dual $\mathfrak{g}^*$ via nondegenerate metric t_{ij}.

Proposition 1.1. *The left-hand sides* $\mathscr{F} = -\mathrm{d}_h\alpha + \frac{1}{2}[\alpha,\alpha]$ *of Maurer–Cartan's equation satisfy the Noether identity* (*or* Bianchi identity *for the curvature two-form*)

$$\partial_\alpha^\dagger(\mathscr{F}) = -\mathrm{d}_h\mathscr{F} - [\mathscr{F},\alpha] \equiv 0. \tag{1.3}$$

Proof. Applying the operator $\partial_\alpha^\dagger$ to the left-hand sides of Maurer–Cartan's equation, we obtain

$$\begin{aligned}\partial_\alpha^\dagger(\mathscr{F}) = \partial_\alpha^\dagger\big(-\mathrm{d}_h\alpha + \tfrac{1}{2}[\alpha,\alpha]\big) = (-\mathrm{d}_h - [\cdot,\alpha])\big(-\mathrm{d}_h\alpha + \tfrac{1}{2}[\alpha,\alpha]\big) = \\ = (\mathrm{d}_h\circ\mathrm{d}_h)\alpha - \tfrac{1}{2}\mathrm{d}_h\big([\alpha,\alpha]\big) + [\mathrm{d}_h\alpha,\alpha] - \tfrac{1}{2}[\alpha,[\alpha,\alpha]] = \\ = -[\mathrm{d}_h\alpha,\alpha] + [\mathrm{d}_h\alpha,\alpha] - \tfrac{1}{2}[\alpha,[\alpha,\alpha]] = 0.\end{aligned}$$

The third term in the last line is zero due to the Jacobi identity, whereas the first two cancel out. □

Let $n = 2$. The Maurer–Cartan equation's left-hand sides $\mathscr{F}$ are top-degree forms, hence every operator which increases the form degree vanishes at $\mathscr{F}$.

Consider the case $n = 3$; we recall that Maurer–Cartan equation (1.1) is Euler–Lagrange in this setup (cf. [1,2,40]).

Proposition 1.2. *If the base manifold* M^3 *is* 3*-dimensional, then Maurer–Cartan's equation is Euler–Lagrange with respect to the action functional*

$$S_{\mathrm{MC}} = \int \mathscr{L} = \int \big\{-\tfrac{1}{2}\langle\alpha,\mathrm{d}_h\alpha\rangle + \tfrac{1}{6}\langle\alpha,[\alpha,\alpha]\rangle\big\}. \tag{1.4}$$

Note that its Lagrangian density $\mathscr{L}$ *is a well-defined top-degree form on the base threefold* M^3.

Proof. Let us construct the Euler–Lagrange equation:

$$\begin{aligned}\delta\int\big\{-\tfrac{1}{2}\langle\alpha,\mathrm{d}_h\alpha\rangle + \tfrac{1}{6}\langle\alpha,[\alpha,\alpha]\rangle\big\} = \langle\delta\alpha,-\mathrm{d}_h\alpha\rangle + \tfrac{1}{6}(\langle\delta\alpha,[\alpha,\alpha]\rangle + \langle\alpha,[\delta\alpha,\alpha]\rangle + \langle\alpha,[\alpha,\delta\alpha]\rangle \\ = \langle\delta\alpha,-\mathrm{d}_h\alpha + \tfrac{1}{2}[\alpha,\alpha]\rangle.\end{aligned}$$

This proves our claim. □

Proposition 1.3. *For each* $p \in \mathfrak{g}\otimes\Lambda^0(M^3)$*, the evolutionary vector field* $\vec{\partial}^{(\alpha)}_{A(p)}$ *with generating section* $A(p) = \partial_\alpha(p) = \mathrm{d}_h p + [p,\alpha]$ *is a Noether symmetry of the action* S_{MC}*,*[d]

$$\vec{\partial}^{(\alpha)}_{A(p)}(S_{\mathrm{MC}}) \cong 0 \in \overline{H}^n(\chi).$$

The operator $A = \partial_\alpha = \mathrm{d}_h + [\cdot,\alpha]$ *determines linear Noether's identity* (1.3)*,*

$$\Phi(x,\alpha,\mathscr{F}) = A^\dagger(\mathscr{F}) \equiv 0,$$

for left-hand sides of the system of Maurer–Cartan's equations (1.1).

Proof. We have

$$\vec{\partial}^{(\alpha)}_{A(p)}S_{\mathrm{MC}} \cong \langle A(p),\tfrac{\delta}{\delta\alpha}S_{\mathrm{MC}}\rangle \cong \Big\langle \big(\ell_\Phi^{(\mathscr{F})}\big)^\dagger(p),\mathscr{F}\Big\rangle \cong \langle p,\ell_\Phi^{(\mathscr{F})}(\mathscr{F})\rangle = \langle p,\Phi(\mathscr{F})\rangle = \langle p,A^\dagger(\mathscr{F})\rangle.$$

In Proposition 1.1 we prove that $A^\dagger(\mathscr{F}) \equiv 0$. So for all p we have that $\langle p,A^\dagger(\mathscr{F})\rangle \cong 0$, which concludes the proof. □

[d]Here $\cong$ denotes the equality up to integration by parts and we assume the absence of boundary terms.

Finally, we let $n > 3$. In this case of higher dimension, the Lagrangian $\mathscr{L} = \langle \alpha, \frac{1}{6}[\alpha,\alpha] - \frac{1}{2}\mathrm{d}_h\alpha\rangle \in \Lambda^3(M^n)$ does not belong to the space of top-degree forms and Proposition 1.2 does not hold. However, Noether's identity $\partial_\alpha^\dagger(\mathscr{F}) \equiv 0$ still holds if $n > 3$ according to Proposition 1.1.

2. Non-Abelian variational Lie algebroids

Let $\vec{e}_1, \ldots, \vec{e}_d$ be a basis in the Lie algebra $\mathfrak{g}$. Every $\mathfrak{g}$-valued zero-curvature representation for a given PDE system $\mathscr{E}^\infty$ is then $\alpha = \alpha_i^k \vec{e}_k \, \mathrm{d}x^i$ for some coefficient functions $\alpha_i^k \in C^\infty(\mathscr{E}^\infty)$. Construct the vector bundle $\chi\colon \Lambda^1(M^n) \otimes \mathfrak{g} \to M^n$ and the trivial bundle $\xi\colon M^n \times \mathfrak{g} \to M^n$ with the Lie algebra $\mathfrak{g}$ taken for fibre. Next, introduce the superbundle $\Pi\xi\colon M^n \times \Pi\mathfrak{g} \to M^n$ the total space of which is the same as that of ξ but such that the parity of fibre coordinates is reversed[e] (see Appendix A.2 on p. 24). Finally, consider the Whitney sum $J^\infty(\chi) \times_{M^n} J^\infty(\Pi\xi)$ of infinite jet bundles over the parity-even vector bundle χ and parity-odd $\Pi\xi$.

With the geometry of every $\mathfrak{g}$-valued zero-curvature representation we associate a non-Abelian variational Lie algebroid [25]. Its realization by a homological evolutionary vector field is the differential in the arising gauge cohomology theory (cf. [37] and [2, 18, 25, 29, 33]).

Theorem 2.1. *The parity-odd evolutionary vector field which encodes the non-Abelian variational Lie algebroid structure on the infinite jet superbundle $J^\infty(\chi \times_{M^n} \Pi\xi) \cong J^\infty(\chi) \times_{M^n} J^\infty(\Pi\xi)$ is*

$$Q = \vec{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h b} + \tfrac{1}{2}\vec{\partial}^{(b)}_{[b,b]}, \qquad [Q,Q] = 0 \quad \Longleftrightarrow \quad Q^2 = 0, \tag{2.1}$$

where for each choice of respective indexes,

- *α_μ^k is a parity-even coordinate along fibres in the bundle χ of $\mathfrak{g}$-valued one-forms,*
- *b^k is a parity-odd fibre coordinate in the bundle $\Pi\xi$,*
- *c_{ij}^k is a structure constant in the Lie algebra $\mathfrak{g}$ so that $[b^i, b^j]^k = b^i c_{ij}^k b^j$ and $[b^i, \alpha^j]^k = b^i c_{ij}^k \alpha^j$,*
- *d_h is the horizontal differential on the Whitney sum of infinite jet bundles,*
- *the operator $\partial_\alpha = \mathrm{d}_h + [\cdot, \alpha]\colon \overline{J^\infty_\chi}(\Pi\xi) \cong J^\infty(\chi \times_{M^n} \Pi\xi) \to \overline{J^\infty_{\Pi\xi}}(\chi) \cong J^\infty(\chi \times_{M^n} \Pi\xi)$ is the anchor.*

Proof. The anticommutator $[Q,Q] = 2Q^2$ of the parity-odd vector field Q with itself is again an evolutionary vector field. Therefore it suffices to prove that the coefficients of $\vec{\partial}/\partial\alpha$ and $\vec{\partial}/\partial b$ are equal to zero in the vector field

$$Q^2 = \left(\vec{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h b} + \tfrac{1}{2}\vec{\partial}^{(b)}_{[b,b]}\right)\left(\vec{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h b} + \tfrac{1}{2}\vec{\partial}^{(b)}_{[b,b]}\right).$$

We have $[b,b]^k = b^i c_{ij}^k b^j$ by definition. Hence it is readily seen that $(\frac{1}{2}\vec{\partial}^{(b)}_{b^i c_{ij}^k b^j})^2 = 0$ because $\mathfrak{g}$ is a Lie algebra [39] so that the Jacobi identity is satisfied by the structure constants. Since the bracket

[e]The odd neighbour $\Pi\mathfrak{g}$ of the Lie algebra is introduced in order to handle poly-linear, totally skew-symmetric maps of elements of $\mathfrak{g}$ so that the parity-odd space $\Pi\mathfrak{g}$ carries the information about the Lie algebra's structure constants c_{ij}^k still not itself becoming a Lie superalgebra.

$[b,b]$ does not depend on α, we deduce that $(\vec{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h b})(\frac{1}{2}\vec{\partial}^{(b)}_{[b,b]}) = 0$. Therefore,

$$Q^2 = \Big(\vec{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h b} + \tfrac{1}{2}\vec{\partial}^{(b)}_{[b,b]}\Big)\Big(\vec{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h b}\Big) = -\vec{\partial}^{(\alpha)}_{[b,[b,\alpha]+\mathrm{d}_h b]} + \tfrac{1}{2}\vec{\partial}^{(\alpha)}_{[[b,b],\alpha]+\mathrm{d}_h([b,b])}$$
$$= \vec{\partial}^{(\alpha)}_{-[b,[b,\alpha]+\mathrm{d}_h b]+\frac{1}{2}[[b,b],\alpha]+\frac{1}{2}\mathrm{d}_h([b,b])}.$$

Now consider the expression $-[b,[b,\alpha]+\mathrm{d}_h b]+\frac{1}{2}[[b,b],\alpha]+\frac{1}{2}\mathrm{d}_h([b,b])$, viewing it as a bi-linear skew-symmetric map $\Gamma(\xi)\times\Gamma(\xi)\to\Gamma(\chi)$. First, we claim that the value $\big(\frac{1}{2}[[b,b],\alpha]-[b,[b,\alpha]]\big)(p_1,p_2)$ at any two sections $p_1, p_2 \in \Gamma(\xi)$ vanishes identically. Indeed, by taking an alternating sum over the permutation group of two elements we have that

$$\tfrac{1}{2}[[p_1,p_2],\alpha]-\tfrac{1}{2}[[p_2,p_1],\alpha]-[p_1,[p_2,\alpha]]+[p_2,[p_1,\alpha]] = [[p_1,p_2],\alpha]-[p_1,[p_2,\alpha]]-[p_2,[\alpha,p_1]]$$
$$= -[\alpha,[p_1,p_2]]-[p_1,[p_2,\alpha]]-[p_2,[\alpha,p_1]=0.$$

At the same time, the value of bi-linear skew-symmetric mapping $\frac{1}{2}\mathrm{d}_h([b,b])-[b,\mathrm{d}_h b]$ at sections p_1 and p_2 also vanishes,

$$\tfrac{1}{2}\mathrm{d}_h([p_1,p_2])-\tfrac{1}{2}\mathrm{d}_h([p_2,p_1])-[p_1,\mathrm{d}_h p_2]+[p_2,\mathrm{d}_h p_1] = \mathrm{d}_h([p_1,p_2])-[p_1,\mathrm{d}_h p_2]-[\mathrm{d}_h p_1,p_2]=0.$$

We conclude that

$$Q^2\Big|_{(p_1,p_2)} = \vec{\partial}^{(\alpha)}_{\{-[b,[b,\alpha]+\mathrm{d}_h b]+\frac{1}{2}[[b,b],\alpha]+\frac{1}{2}\mathrm{d}_h([b,b])\}(p_1,p_2)} = \vec{\partial}^{(\alpha)}_0 = 0,$$

which proves the theorem. □

Finally, let us derive a reparametrization formula for the homological vector field Q in the course of gauge transformations of zero-curvature representations. We begin with some trivial facts [7,11].

Lemma 2.1. *Let α be a $\mathfrak{g}$-valued zero-curvature representation for a PDE system. Consider two infinitesimal gauge transformations given by $g_1 = 1+\varepsilon p_1 + o(\varepsilon)$ and $g_2 = 1+\varepsilon p_2 + o(\varepsilon)$. Let $g \in C^\infty(\mathscr{E}^\infty, G)$ also determine a gauge transformation. Then the following diagram is commutative,*

$$\begin{array}{ccc} \alpha^g & \xrightarrow{g_2} & \beta \\ \uparrow g & & \uparrow g \\ \alpha & \xrightarrow{g_1} & \alpha^{g_1}, \end{array}$$

if the relation $p_2 = g\cdot p_1\cdot g^{-1}$ is valid.

Proof. By the lemma's assumption we have that $(\alpha^{g_1})^g = (\alpha^g)^{g_2}$. Hence we deduce that

$$g\cdot(1+\varepsilon p_1) = (1+\varepsilon p_2)\cdot g \qquad \Longleftrightarrow \qquad g\cdot p_1 = p_2\cdot g,$$

which yields the transformation rule $p_2 = g\cdot p_1\cdot g^{-1}$ for the $\mathfrak{g}$-valued function p_1 on $\mathscr{E}^\infty$ in the course of gauge transformation $g\colon \alpha \mapsto \alpha^g$. □

Using the above lemma we describe the behaviour of homological vector field Q in the non-Abelian variational setup of Theorem 2.1.

Corollary 2.1. *Under a coordinate change*

$$\alpha \mapsto \alpha' = g\cdot\alpha\cdot g^{-1} + d_h g\cdot g^{-1}, \qquad b\mapsto b' = g\cdot b\cdot g^{-1},$$

where $g\in C^\infty(M^n,G)$*, the variational Lie algebroid's differential* Q *is transformed accordingly*:

$$Q \longmapsto Q' = \vec{\partial}^{(\alpha')}_{[b',\alpha']+\mathrm{d}_h b'} + \tfrac{1}{2}\vec{\partial}^{(b')}_{[b',b']}.$$

3. The master-functional for zero-curvature representations

The correspondence between zero-curvature representations, i.e., classes of gauge-equivalent solutions α to the Maurer–Cartan equation, and non-Abelian variational Lie algebroids goes in parallel with the BRST-technique, in the frames of which ghost variables appear and gauge algebroids arise (see [3, 22]). Let us therefore extend the BRST-setup of fields α and ghosts b to the full BV-zoo of (anti)fields α and α^* and (anti)ghosts b and b^* (cf. [4, 5, 15]). We note that a finite-dimensional 'forefather' of what follows is discussed in detail in [2], which is devoted to Q- and QP-structures on (super)manifolds. Those concepts are standard; our message is that not only the approach of [2] to QP-structures on G-manifolds X and $\Pi T^*\bigl(X\times\Pi TG/G\bigr)\simeq \Pi T^*X\times\mathfrak{g}^*\times\Pi\mathfrak{g}$ remains applicable in the variational setup of jet bundles (i.e., whenever integrations by parts are allowed, whence many Leibniz rule structures are lost, see Appendix A), but even the explicit formulas for the BRST-field Q and the action functional $\widehat{S}$ for the extended field $\widehat{Q}$ are valid literally. In fact, we recover the *third* and *fourth* equivalent formulations of the definition for a variational Lie algebroid (cf. [2, 37] or a review [30]).

Let us recall from section 2 that α is a tuple of even-parity fibre coordinates in the bundle $\chi\colon \Lambda^1(M^n)\otimes\mathfrak{g}\to M^n$ and b are the odd-parity coordinates along fibres in the trivial vector bundle $\Pi\xi\colon M^n\times\Pi\mathfrak{g}\to M^n$. We now let all the four *neighbours* of the Lie algebra $\mathfrak{g}$ appear on the stage: they are $\mathfrak{g}$ (in χ), $\mathfrak{g}^*$, $\Pi\mathfrak{g}$ (in $\Pi\xi$), and $\Pi\mathfrak{g}^*$ (see [39] and reference therein). Let us consider the bundle $\Pi\chi^*\colon\ \mathrm{D}_1(M^n)\otimes\Pi\mathfrak{g}^*\to M^n$ whose fibres are dual to those in χ and also have the parity reversed.[f] We denote by α^* the collection of odd fibre coordinates in $\Pi\chi^*$.

Remark 3.1. In what follows we do not write the (indexes for) bases of vectors in the fibres of $\mathrm{D}_1(M^n)$ or of covectors in $\Lambda^1(M^n)$; to make the notation short, their couplings are implicit. Nevertheless, a summation over such "invisible" indexes in $\partial/\partial x^\mu$ and $\mathrm{d}x^\nu$ is present in all formulas containing the couplings of α and α^*. We also note that $(\alpha^*)\overleftarrow{\mathrm{d}_h}$ is a very interesting object because α^* parametrizes fibres in $\mathrm{D}_1(M^n)\otimes\Pi\mathfrak{g}^*$; the horizontal differential d_h produces the forms $\mathrm{d}x^i$ which are initially not coupled with their duals from $\mathrm{D}_1(M^n)$. (However, such objects cancel out in the identity $\widehat{Q}^2=0$, see (3.4) on p. 13.)

Secondly, we consider the even-parity dual $\xi^*\colon M^n\times\mathfrak{g}^*\to M^n$ of the odd bundle $\Pi\xi$; let us denote by b^* the coordinates along $\mathfrak{g}^*$ in the fibres of ξ^*.

[f] In terms of [2], the Whitney sum $J^\infty(\chi)\times_{M^n}J^\infty(\Pi\chi^*)$ plays the rôle of ΠT^*X for a G-manifold X; here $\mathfrak{g}$ is the Lie algebra of a Lie group G so that $\Pi\mathfrak{g}\simeq\Pi TG/G$.

Finally, we fix the ordering

$$\delta\alpha\wedge\delta\alpha^{*}+\delta b^{*}\wedge\delta b \tag{3.1}$$

of the canonically conjugate pairs of coordinates. By picking a volume form $\mathrm{dvol}(M^n)$ on the base M^n we then construct the odd Poisson bracket (variational Schouten bracket $[\![\,,\,]\!]$) on the senior d_h-cohomology (or *horizontal* cohomology) space $\overline{H}^n(\chi\times_{M^n}\Pi\chi^{*}\times_{M^n}\Pi\xi\times_{M^n}\xi^{*})$; we refer to [20,21] for a geometric theory of variations.

Theorem 3.1. *The structure of non-Abelian variational Lie algebroid from Theorem* 2.1 *is encoded on the Whitney sum $J^\infty(\chi\times_{M^n}\Pi\chi^{*}\times_{M^n}\Pi\xi\times_{M^n}\xi^{*})$ of infinite jet (super)bundles by the action functional*

$$\widehat{S}=\int\mathrm{dvol}(M^n)\left\{\langle\alpha^{*},[b,\alpha]+\mathrm{d}_h(b)\rangle+\tfrac{1}{2}\langle b^{*},[b,b]\rangle\right\}\in\overline{H}^n(\chi\times_{M^n}\Pi\chi^{*}\times_{M^n}\Pi\xi\times_{M^n}\xi^{*})$$

which satisfies the classical master-equation

$$[\![\widehat{S},\widehat{S}]\!]=0.$$

The functional $\widehat{S}$ is the Hamiltonian of odd-parity evolutionary vector field $\widehat{Q}$ which is defined on $J^\infty(\chi)\times_{M^n}J^\infty(\Pi\chi^{*})\times_{M^n}J^\infty(\Pi\xi)\times_{M^n}J^\infty(\xi^{*})$ by the equality

$$\widehat{Q}(\mathscr{H})\cong[\![\widehat{S},\mathscr{H}]\!] \tag{3.2}$$

for any $\mathscr{H}\in\overline{H}^n(\chi\times_{M^n}\Pi\chi^{*}\times_{M^n}\Pi\xi\times_{M^n}\xi^{*})$. The odd-parity field is[g]

$$\widehat{Q}=\vec{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h(b)}+\vec{\partial}^{(\alpha^{*})}_{(\alpha^{*})\overleftarrow{\mathrm{ad}}^{*}_{b}}+\tfrac{1}{2}\vec{\partial}^{(b)}_{[b,b]}+\vec{\partial}^{(b^{*})}_{-\mathrm{ad}^{*}_{\alpha}(\alpha^{*})+(\alpha^{*})\overleftarrow{\mathrm{d}_h}+\mathrm{ad}^{*}_{b}(b^{*})}, \tag{3.3}$$

where $\langle(\alpha^{*})\overleftarrow{\mathrm{ad}}^{*}_{b},\alpha\rangle\stackrel{\text{def}}{=}\langle\alpha^{*},[b,\alpha]\rangle$ and $\langle\mathrm{ad}^{*}_{b}(b^{*}),p\rangle=\langle b^{*},[b,p]\rangle$ for any $\alpha\in\Gamma(\chi)$ and $p\in\Gamma(\xi)$. This evolutionary vector field is homological,

$$\widehat{Q}^2=0.$$

Proof. In coordinates, the master-action $\widehat{S}=\int\widehat{\mathscr{L}}\,\mathrm{dvol}(M^n)$ is equal to

$$\widehat{S}=\int\mathrm{dvol}(M^n)\left\{\alpha^{*}_{a}(b^{\mu}c^{a}_{\mu\nu}\alpha^{\nu}+\mathrm{d}_h(b^{a}))+\tfrac{1}{2}b^{*}_{\mu}b^{\beta}c^{\mu}_{\beta\gamma}b^{\gamma}\right\};$$

here the summation over spatial degrees of freedom from the base M^n in implicit in the horizontal differential d_h and the respective contractions with α^{*}. By the Jacobi identity for the variational Schouten bracket $[\![\,,\,]\!]$ (see [21]), the classical master equation $[\![\widehat{S},\widehat{S}]\!]=0$ is equivalent to the homological condition $\widehat{Q}^2=0$ for the odd-parity vector field defined by (3.2). The conventional choice

[g] The referee points out that the evolutionary vector field $\widehat{Q}$ is the jet-bundle upgrade of the *cotangent lift* of the field Q, which is revealed by the explicit formula for the Hamiltonian $\widehat{S}$. Let us recall that the cotangent lift of a vector field $\mathscr{Q}=\mathscr{Q}^i\,\partial/\partial q^i$ on a (super)manifold N^m is the Hamiltonian vector field on $T^{*}N^m$ given by $\widehat{\mathscr{Q}}=\mathscr{Q}^i(q)\,\partial/\partial q^i-p_j\cdot\partial\mathscr{Q}^j(q)/\partial q^i\,\partial/\partial p_i$; its Hamiltonian is $\mathscr{S}=p_i\,\mathscr{Q}^i(q)$. An example of this classical construction is contained in the seminal paper [2].

of signs (3.1) yields a formula for this graded derivation,

$$\widehat{Q}=\vec{\partial}^{(\alpha)}_{-\vec{\delta}\widehat{\mathscr{L}}/\delta\alpha^*}+\vec{\partial}^{(\alpha^*)}_{\vec{\delta}\widehat{\mathscr{L}}/\delta\alpha}+\vec{\partial}^{(b)}_{\vec{\delta}\widehat{\mathscr{L}}/\delta b^*}+\vec{\partial}^{(b^*)}_{-\vec{\delta}\widehat{\mathscr{L}}/\delta b},$$

where the arrows over $\vec{\partial}$ and $\vec{\delta}$ indicate the direction along which the graded derivations act and graded variations are transported (that is, from left to right and rightmost, respectively). We explicitly obtain that[h]

$$\widehat{Q}=\vec{\partial}^{(\alpha^a)}_{b^\mu c^a_{\mu\nu}\alpha^\nu+\mathrm{d}_h(b^a)}+\vec{\partial}^{(\alpha^*_\nu)}_{\alpha^*_a b^\mu c^a_{\mu\nu}}+\vec{\partial}^{(b^\mu)}_{\frac{1}{2}b^\beta c^\mu_{\beta\gamma}b^\gamma}+\vec{\partial}^{(b^*_\mu)}_{\left\{-\alpha^*_a c^a_{\mu\nu}\alpha^\nu+(\alpha^*_\mu)\overleftarrow{\mathrm{d}_h}+b^*_a c^a_{\mu\nu}b^\nu\right\}}.$$

Actually, the proof of Theorem 2.1 contains the first half of a reasoning which shows why $\widehat{Q}^2=0$. (It is clear that the field $\widehat{Q}$ consists of (2.1) not depending on α^* and b^* and of the two new terms.) Again, the anticommutator $[\widehat{Q},\widehat{Q}]=2\widehat{Q}^2$ is an evolutionary vector field. We claim that the coefficients of $\vec{\partial}/\partial\alpha^*_\nu$ and $\vec{\partial}/\partial b^*_\mu$ in it are equal to zero.

Let us consider first the coefficient of $\vec{\partial}/\partial\alpha^*$ at the bottom of the evolutionary derivation $\vec{\partial}^{(\alpha^*)}_{\{\dots\}}$ in $\widehat{Q}^2$; by contracting this coefficient with $\alpha=(\alpha^\nu)$ we obtain

$$\langle\alpha^*_a, b^\lambda c^a_{\lambda\mu}b^q c^\mu_{q\nu}\alpha^\nu-\tfrac{1}{2}b^\beta c^\mu_{\beta\gamma}b^\gamma c^a_{\mu\nu}\alpha^\nu\rangle.$$

It is readily seen that α^* is here coupled with the bi-linear skew-symmetric operator $\Gamma(\xi)\times\Gamma(\xi)\to\Gamma(\chi)$ for any fixed $\alpha\in\Gamma(\chi)$, and we show that this operator is zero on its domain of definition. Indeed, the comultiple $|\rangle$ of $\langle\alpha^*|$ is $[b,[b,\alpha]]-\frac{1}{2}[[b,b],\alpha]$ so that its value at any arguments $p_1,p_2\in\Gamma(\xi)$ equals

$$[p_1,[p_2,\alpha]]-[p_2,[p_1,\alpha]]-[\tfrac{1}{2}[p_1,p_2]-\tfrac{1}{2}[p_2,p_1],\alpha]=0$$

by the Jacobi identity.

Let us now consider the coefficient of $\vec{\partial}/\partial b^*_\mu$ in the vector field $\widehat{Q}^2$,

$$-\left[\alpha^*_{\tilde{a}}b^{\tilde{\mu}}c^{\tilde{a}}_{\tilde{\mu}a}\right]c^a_{\mu\nu}\alpha^\nu+\alpha^*_a c^a_{\mu\nu}\left[b^{\tilde{\mu}}c^\nu_{\tilde{\mu}\tilde{\nu}}\alpha^{\tilde{\nu}}+\mathrm{d}_h(b^\nu)\right]+\left(\left[\alpha^*_{\tilde{a}}b^{\tilde{\mu}}c^{\tilde{a}}_{\tilde{\mu}\mu}\right]\right)\overleftarrow{\mathrm{d}_h}$$
$$+\left[-\alpha^*_{\tilde{a}}c^{\tilde{a}}_{a\tilde{\nu}}\alpha^{\tilde{\nu}}+(\alpha^*_a)\overleftarrow{\mathrm{d}_h}+b^*_{\tilde{a}}c^{\tilde{a}}_{a\tilde{\nu}}b^{\tilde{\nu}}\right]c^a_{\mu\nu}b^\nu+b^*_a c^a_{\mu\nu}\cdot\left[\tfrac{1}{2}b^{\tilde{\beta}}c^\nu_{\tilde{\beta}\tilde{\gamma}}b^{\tilde{\gamma}}\right];$$

here we mark with a tilde sign those summation indexes which come from the first copy of $\widehat{Q}$ acting from the left on $\vec{\partial}^{(b^*_\mu)}_{\{\dots\}}$ in $\widehat{Q}\circ\widehat{Q}$. Two pairs of cancellations occur in the terms which contain the horizontal differential d_h. First, let us consider the terms in which the differential acts on α^*. By

[h]Note that $\langle\alpha^*,\overrightarrow{\mathrm{d}_h}(b)\rangle\cong-\langle(\alpha^*)\overleftarrow{\mathrm{d}_h},b\rangle$ in the course of integration by parts, whence the term $(\alpha^*_\mu)\overleftarrow{\mathrm{d}_h}$ that comes from $-\vec{\delta}\widehat{\mathscr{L}}/\delta b^\mu$ does stand with a plus sign in the velocity of b^*_μ.

contracting the index μ with an extra copy $b = (b^\mu)$, we obtain

$$(\alpha_a^*)\overleftarrow{\mathrm{d}_h}\ b^\lambda c^a_{\lambda\mu} b^\mu + (\alpha_a^*)\overleftarrow{\mathrm{d}_h}\ c^a_{\mu\lambda} b^\lambda b^\mu. \tag{3.4}$$

Due to the skew-symmetry of structure constants c^k_{ij} in $\mathfrak{g}$, at any sections $p_1, p_2 \in \Gamma(\xi)$ we have that

$$(\alpha_a^*)\overleftarrow{\mathrm{d}_h} \cdot \left(p_1^\lambda c^a_{\lambda\mu} p_2^\mu - p_2^\lambda c^a_{\lambda\mu} p_1^\mu + c^a_{\mu\lambda} p_1^\lambda p_2^\mu - c^a_{\mu\lambda} p_2^\lambda p_1^\mu\right) = 0.$$

Likewise, a contraction with $b = (b^\mu)$ for the other pair of terms with d_h, now acting on b, yields

$$\alpha_a^* c^a_{\mu\lambda}\, \mathrm{d}_h(b^\lambda) b^\mu + \alpha_a^*\, \mathrm{d}_h(b^\lambda)\, c^a_{\lambda\mu} b^\mu. \tag{3.5}$$

At the moment of evaluation at p_1 and p_2, expression (3.5) cancels out due to the same mechanism as above.

The remaining part of the coefficient of $\vec{\partial}/\partial b^*_\mu$ in $\widehat{Q}^2$ is

$$\begin{aligned} &-\alpha_z^* b^\lambda c^z_{\lambda a} c^a_{\mu\nu} \alpha^\nu + \alpha_z^* c^z_{\mu\nu} b^i c^\nu_{ij} \alpha^j - \alpha_z^* c^z_{a\nu} \alpha^\nu c^a_{\mu j} b^j \\ &\qquad + b^*_\lambda c^\lambda_{a\gamma} b^\gamma c^a_{\mu j} b^j + b^*_\lambda c^\lambda_{\mu\gamma} \cdot \tfrac{1}{2} b^\beta c^\gamma_{\beta\delta} b^\delta. \end{aligned} \tag{3.6}$$

It is obvious that the mechanisms of vanishing are different for the first and second lines in (3.6) whenever each of the two is regarded as mapping which takes $b = (b^\mu)$ to a number from the field $\Bbbk$. Therefore, let us consider these two lines separately.

By contracting the upper line of (3.6) with $b = (b^\mu)$, we rewrite it as follows,

$$\langle -\alpha_z^*, b^\lambda c^z_{\lambda a} c^a_{\mu\nu} \alpha^\nu b^\mu - c^z_{\mu\nu} b^i c^\nu_{ij} \alpha^j b^\mu + c^z_{a\nu} \alpha^\nu c^a_{\mu j} b^j b^\mu \rangle.$$

Viewing the content of the co-multiple $|\rangle$ of $\langle -\alpha^*|$ as bi-linear skew-symmetric mapping $\Gamma(\xi) \times \Gamma(\xi) \to \Gamma(\chi)$, we conclude that its value at any pair of section $p_1, p_2 \in \Gamma(\xi)$ is

$$\begin{aligned} &[p_2, [p_1, \alpha]] - [p_1, [p_2, \alpha]] + [[p_1, p_2], \alpha] \\ &- [p_1, [p_2, \alpha]] + [p_2, [p_1, \alpha]] - [[p_2, p_1], \alpha] = 0 - 0 = 0, \end{aligned}$$

because each line itself amounts to the Jacobi identity.

At the same time, the contraction of lower line in (3.6) with $b = (b^\mu)$ gives

$$\langle b^*_\lambda, c^\lambda_{a\gamma} b^\gamma c^a_{\mu j} b^j b^\mu + c^\lambda_{\mu\gamma} \cdot \tfrac{1}{2} b^\beta c^\gamma_{\beta\delta} b^\delta b^\mu \rangle.$$

The term $|\rangle$ near $\langle b^*|$ determines the tri-linear skew-symmetric mapping $\Gamma(\xi) \times \Gamma(\xi) \times \Gamma(\xi) \to \Gamma(\xi)$ whose value at any $p_1, p_2, p_3 \in \Gamma(\xi)$ is defined by the formula

$$\sum_{\sigma \in S_3} (-)^\sigma \Big\{ \big[[p_{\sigma(1)}, p_{\sigma(2)}], p_{\sigma(3)}\big] + \big[p_{\sigma(1)}, \tfrac{1}{2}[p_{\sigma(2)}, p_{\sigma(3)}]\big] \Big\}.$$

This amounts to four copies of the Jacobi identity (indeed, let us take separate sums over even and odd permutations). Consequently, the tri-linear operator at hand, hence the entire coefficient of $\vec{\partial}/\partial b^*$, is equal to zero so that $\widehat{Q}^2 = 0$. □

4. Gauge automorphisms of the $\widehat{Q}$-cohomology groups

We finally describe the next generation of Lie algebroids; they arise from infinitesimal gauge symmetries of the quantum master-equation (0.2) or its limit $[\![\widehat{S},\widehat{S}]\!]=0$ as $\hbar\to 0$. The construction of infinitesimal gauge automorphisms illustrates general principles of theory of differential graded Lie- or L_∞-algebras (see [2,29] and [18]).

Theorem 4.1. *An infinitesimal shift $\widehat{S}\mapsto\widehat{S}(\varepsilon)=\widehat{S}+\varepsilon[\![\widehat{S},F]\!]+o(\varepsilon)$, where F is an odd-parity functional, is a gauge symmetry of the classical master-equation $[\![\widehat{S},\widehat{S}]\!]=0$. A simultaneous shift $\eta\mapsto\eta(\varepsilon)=\eta+\varepsilon[\![\eta,F]\!]+o(\varepsilon)$ of all functionals $\eta\in\overline{H}^n(\chi\times_{M^n}\Pi\chi^*\times_{M^n}\Pi\xi\times_{M^n}\xi^*)$, but not of the generator F itself, preserves the structure of $\widehat{Q}$-cohomology classes.*

Proof. Let F be an odd-parity functional and perform the infinitesimal shift $\widehat{S}\mapsto\widehat{S}+\varepsilon[\![\widehat{S},F]\!]+o(\varepsilon)$ of the Hamiltonian $\widehat{S}$ for the differential $\widehat{Q}$. We have that

$$[\![\widehat{S}(\varepsilon),\widehat{S}(\varepsilon)]\!]=[\![\widehat{S},\widehat{S}]\!]+2\varepsilon[\![\widehat{S},[\![\widehat{S},F]\!]]\!]+o(\varepsilon).$$

By using the shifted-graded Jacobi identity for the variational Schouten bracket $[\![\,,\,]\!]$ (see [21]) we deduce that

$$[\![\widehat{S},[\![\widehat{S},F]\!]]\!]=\tfrac{1}{2}[\![[\![\widehat{S},\widehat{S}]\!],F]\!],$$

so that the infinitesimal shift is a symmetry of the classical master-equation $[\![\widehat{S},\widehat{S}]\!]=0$.

Now let a functional η mark a $\widehat{Q}$-cohomology class, i.e., suppose $[\![\widehat{S},\eta]\!]=0$. In the course of simultaneous evolution $\widehat{S}\mapsto\widehat{S}(\varepsilon)$ for the classical master-action and $\eta\mapsto\eta(\varepsilon)$ for $\widehat{Q}$-cohomology elements, the initial condition $[\![\widehat{S},\eta]\!]=0$ at $\varepsilon=0$ evolves as fast as

$$[\![[\![\widehat{S},F]\!],\eta]\!]+[\![\widehat{S},[\![\eta,F]\!]]\!]=[\![[\![\widehat{S},\eta]\!],F]\!]=0$$

due to the Jacobi identity and the cocycle condition itself. In other words, the $\widehat{Q}$-cocycles evolve to $\widehat{Q}(\varepsilon)$-cocycles.

At the same time, let *all* functionals $h\in\overline{H}^n(\chi\times_{M^n}\Pi\chi^*\times_{M^n}\Pi\xi\times_{M^n}\xi^*)$ evolve by the law $h\mapsto h(\varepsilon)=h+\varepsilon[\![h,F]\!]+o(\varepsilon)$. Consider two representatives, η and $\eta+[\![\widehat{S},h]\!]$, of the $\widehat{Q}$-cohomology class for a functional η. On one hand, the velocity of evolution of the $\widehat{Q}$-exact term $[\![\widehat{S},h]\!]$ is postulated to be $[\![[\![\widehat{S},h]\!],F]\!]$; we claim that the infinitesimally shifted functional $[\![\widehat{S},h]\!](\varepsilon)$ remains $\widehat{Q}(\varepsilon)$-exact. Indeed, on the other hand we have that, knowing the change $\widehat{S}\mapsto\widehat{S}(\varepsilon)$ and $h\mapsto h(\varepsilon)$, the exact term's calculated velocity is

$$[\![[\![\widehat{S},F]\!],h]\!]+[\![\widehat{S},[\![h,F]\!]]\!]=[\![[\![\widehat{S},h]\!],F]\!]$$

(the Jacobi identity for $[\![\,,\,]\!]$ works again and the assertion is valid irrespective of the parity of h whenever F is parity-odd). This shows that the postulated and calculated evolutions of $\widehat{Q}$-exact terms coincide, whence $\widehat{Q}$-coboundaries become $\widehat{Q}(\varepsilon)$-coboundaries after the infinitesimal shift. We conclude that the structure of $\widehat{Q}$-cohomology group stays intact under such transformations of the space of functionals. □

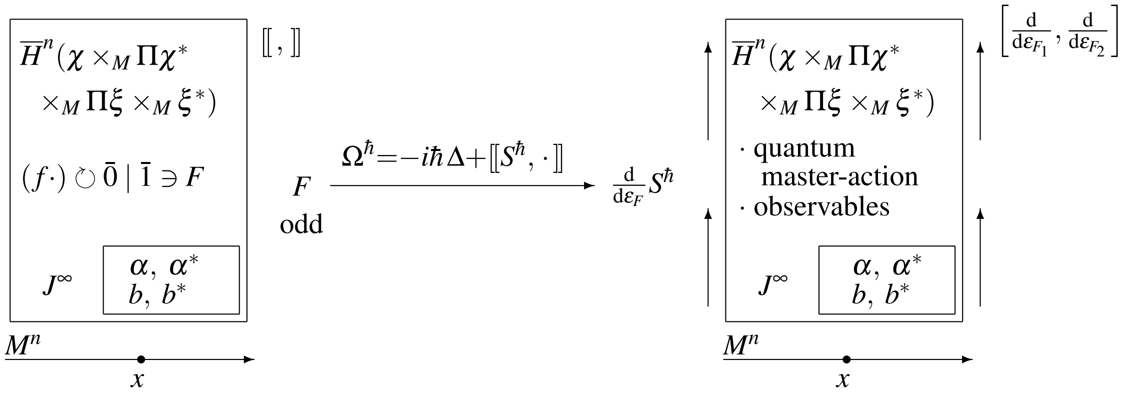

Fig. 2. The next generation of Lie algebroids: gauge automorphisms of the (quantum) BV-cohomology.

The above picture of gauge automorphisms is extended verbatim to the full quantum setup[i], see Fig. 2 and [20, § 3.2] for detail. Ghost parity-odd functionals $F \in \overline{H}^n\,(\chi \times_{M^n} \Pi\chi^* \times_{M^n} \Pi\xi \times_{M^n} \xi^*)$ are the generators of gauge transformations

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon_F} S^\hbar = \Omega^\hbar(F);$$

a parameter $\varepsilon_F \in \mathbb{R}$ is (formally) associated with every odd functional F. The *observables* f arise through expansions $S^\hbar + \lambda f + o(\lambda)$ of the quantum master-action ; their evolution is given by the coefficient

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon_F} f = [\![f, F]\!]$$

of λ in the velocity of full action functional. It is clear also why the evolution of gauge generators F – that belong to the domain of definition of $\Omega^\hbar$ but not to its image – is not discussed at all.

Let us recall from [20, § 3.2] and [21] that the commutator of two infinitesimal gauge transformations with ghost parity-odd parameters, say $\mathcal{X}$ and $\mathcal{Y}$, is determined by the variational Schouten bracket of the two generators :

$$\left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon_{\mathcal{Y}}} \circ \frac{\mathrm{d}}{\mathrm{d}\varepsilon_{\mathcal{X}}} - \frac{\mathrm{d}}{\mathrm{d}\varepsilon_{\mathcal{X}}} \circ \frac{\mathrm{d}}{\mathrm{d}\varepsilon_{\mathcal{Y}}}\right) S^\hbar = \Omega^\hbar\big([\![\mathcal{X}, \mathcal{Y}]\!]\big).$$

Moreover, we discover that parity-even observables f play the rôle of “functions” in the world of formal products of integral functionals. Namely, we have that

$$[\![f\cdot\mathcal{X}, \mathcal{Y}]\!] = f\cdot[\![\mathcal{X}, \mathcal{Y}]\!] - (-)^{|\mathcal{X}|\cdot|\mathcal{Y}|} \frac{\mathrm{d}}{\mathrm{d}\varepsilon_{\mathcal{Y}}}(f)\cdot\mathcal{X}.$$

In these terms, we recover the *classical* notion of Lie algebroid — at the quantum level of horizontal cohomology modulo $\operatorname{im}\mathrm{d}_h$ in the variational setup; that classical concept is reviewed in Appendix A.1, see p. 22. The new Lie algebroid is encoded by

[i] The Batalin–Vilkovisky differential $\Omega^\hbar$ stems from the Schwinger–Dyson condition of effective independence – of the ghost parity-odd degrees of freedom – for Feynman’s path integrals of the observables ; in earnest, the condition expresses the intuitive property $\langle 1\rangle = 1$ of averaging with weight factor $\exp\left(\frac{i}{\hbar}S^\hbar\right)$, see [4, 15] and [20].

- the parity-odd part of the superspace $\overline{H}^n(\chi \times_{M^n} \Pi\chi^* \times_{M^n} \Pi\xi \times_{M^n} \xi^*)$ fibred over the infinite jet space for the Whitney sum of bundles (cf. Remark 1.1 on p. 3);
- the quantum Batalin–Vilkovisky differential $\Omega^\hbar$, which is the anchor ;
- the Schouten bracket $[\![\,,\,]\!]$, which is the Lie (super)algebra structure on the infinite-dimensional, parity-odd homogeneity component of $\overline{H}^n(\chi \times_{M^n} \Pi\chi^* \times_{M^n} \Pi\xi \times_{M^n} \xi^*)$ containing the generators of gauge automorphisms in the quantum BV-model at hand.

We see that the link between the BV-differential $\Omega^\hbar$ and the classical Lie algebroid in Fig. 2 is exactly the same as the relationship between Marvan's operator ∂_α and the non-Abelian variational Lie algebroid in Fig. 1.

Let us conclude this paper by posing an open problem of realization of the newly-built classical Lie algebroid via the master-functional $\mathcal{S}$ and Schouten bracket in the bi-graded, infinite-dimensional setup over the superbundle of ghost parity-odd generators of gauge automorphisms (see Theorem 4.1). We further the question to the problem of deformation quantization in the geometry of that classical master-equation for $\mathcal{S}$, see [28]. The difficulty which should be foreseen at once is that the cohomological deformation technique (see [4, 28] or [15, 38] and references therein) is known to be not always valid in the infinite dimension. A successful solution of the deformation quantization problem – or its (re-)iterations at higher levels, much along the lines of this paper – will yield the deformation parameter(s) which would be different from $\hbar$; for the Planck constant is engaged already in the picture. On the other hand, a rigidity statement would show that there can be no deformation parameters beyond the Planck constant $\hbar$.

Conclusion

Let us sum up the geometries we are dealing with. We started with a partial differential equation $\mathcal{E}$ for physical fields; it is possible that $\mathcal{E}$ itself was Euler–Lagrange[j] and it could be gauge-invariant with respect to some Lie group. We then recalled the notion of $\mathfrak{g}$-valued zero-curvature representations α for $\mathcal{E}$; here $\mathfrak{g}$ is the Lie algebra of a given Lie group G and α is a flat connection's 1-form in a principal G-bundle over $\mathcal{E}^\infty$. By construction, this $\mathfrak{g}$-valued horizontal form satisfies the Maurer–Cartan equation

$$\mathcal{E}_{\mathrm{MC}} = \left\{ \mathrm{d}_h\alpha \doteq \tfrac{1}{2}[\alpha,\alpha] \right\} \tag{1.1}$$

by virtue of $\mathcal{E}$ and its differential consequences which constitute $\mathcal{E}^\infty$. System (1.1) is always gauge-invariant so that there are linear Noether's identities (1.3) between the equations; if the base manifold M^n is three-dimensional, then the Maurer–Cartan equation $\mathcal{E}_{\mathrm{MC}}$ is Euler–Lagrange with respect to action functional (1.4). The main result of this paper (see Theorem 3.1 on p. 11) is that – whenever one takes not just the bundle χ for $\mathfrak{g}$-valued 1-forms but the Whitney sum of four (infinite jet bundles over) vector bundles with prototype fibers built from $\mathfrak{g}$, $\Pi\mathfrak{g}$, $\mathfrak{g}^*$, and $\Pi\mathfrak{g}^*$ – the gauge invariance in (1.1) is captured by evolutionary vector field (3.3) with Hamiltonian $\widehat{S}$ that satisfies the classical

[j]The class of admissible models is much wider than it may first seem; for example, the Korteweg–de Vries equation $w_t = -\frac{1}{2}w_{xxx} + 3ww_x$ is Euler–Lagrange with respect to the action functional $S_0 = \int \left\{ \frac{1}{2}v_xv_t - \frac{1}{4}v_{xx}^2 - \frac{1}{2}v_x^3 \right\} \mathrm{d}x\wedge\mathrm{d}t$ if one sets $w = v_x$. In absence of the model's own gauge group, its BV-realization shrinks but there remains gauge invariance in the Maurer–Cartan equation.

master-equation [2, 13],

$$\mathscr{E}_{\text{CME}} = \big\{ i\hbar\Delta\widehat{S}\big|_{\hbar=0} = \tfrac{1}{2}[\![\widehat{S},\widehat{S}]\!]\big\}. \tag{0.1}$$

We notice that, by starting with the geometry of solutions to Maurer–Cartan's equation (1.1), we have constructed another object in the category of differential graded Lie algebras [29]; namely, we arrive at a setup with *zero* differential $i\hbar\Delta\big|_{\hbar=0}$ and Lie (super-)algebra structure defined by the variational Schouten bracket $[\![\,,\,]\!]$. That geometry's genuine differential at $\hbar \neq 0$ is given by the Batalin–Vilkovisky Laplacian Δ (see [4] and [20] for its definition). Let us now examine whether the standard BV-technique ([4, 15], cf. [8]) can be directly applied to the case of zero-curvature representations, hence to quantum inverse scattering ([35] and [32], also [10, 12]).

It is obvious that the equations of motion $\mathscr{E}$ upon physical fields $u = \phi(x)$ co-exist with the Maurer-Cartan equations satisfied by zero-curvature representations α. The geometries of non-Abelian variational Lie algebroids and gauge algebroids [3, 22] are two manifestations of the same construction; let us stress that the respective gauge groups can be unrelated: there is the Lie group G for $\mathfrak{g}$-valued zero-curvature representations α and, on the other hand, there is a gauge group (if any, see footnote j) for physical fields and their equations of motion $\mathscr{E} = \{\delta S_0/\delta u = 0\}$.

We recalled in section 1.4 that the Maurer–Cartan equation $\mathscr{E}_{\text{MC}}$ itself is Euler–Lagrange with respect to functional (1.4) in the class of bundles over threefolds, cf. [1, 2, 40]. One obtains the Batalin–Vilkovisky action by extending the geometry of zero-curvature representations in order to capture Noether's identities (1.3). It is readily seen that the required set of Darboux variables consists of

- the coordinates $\mathscr{F}$ along fibres in the bundle $\mathfrak{g}^* \otimes \Lambda^2(M^3)$ for the equations $\mathscr{E}_{\text{MC}}$,
- the *antifields* $\mathscr{F}^\dagger$ for the bundle $\Pi\mathfrak{g} \otimes \Lambda^1(M^3)$ which is dual to the former and which has the opposite $\mathbb{Z}_2$-valued ghost parity,[k] and also
- the *antighosts* $b^\dagger$ along fibres of $\mathfrak{g}^* \otimes \Lambda^3(M^3)$ which reproduce syzygies (1.3), as well as
- the *ghosts* b from the dual bundle $\Pi\mathfrak{g} \times M^3 \to M^3$.

The standard Koszul–Tate term in the Batalin–Vilkovisky action is then $\langle b, \partial^\dagger_\alpha(\alpha^\dagger)\rangle$: the classical master-action for the entire model is then[l]

$$(S_0 + \langle\text{BV-terms}\rangle) + (S_{\text{MC}} + \langle\text{Koszul-Tate}\rangle);$$

the respective BV-differentials anticommute in the Whitney sum of the two geometries for physical fields and flat connection $\mathfrak{g}$-forms.

The point is that Maurer–Cartan's equation (1.1) is Euler–Lagrange only if $n = 3$; however, the system $\mathscr{E}_{\text{MC}}$ remains gauge invariant at all $n \geqslant 2$ but the attribution of (anti)fields and (anti)ghosts to the bundles as above becomes *ad hoc* if $n \neq 3$. We therefore propose to switch from the BV-approach to a picture which employs the four neighbours $\mathfrak{g}$, $\Pi\mathfrak{g}$, $\mathfrak{g}^*$, and $\Pi\mathfrak{g}^*$ within the master-action $\widehat{S}$. This

[k] The co-multiple $|\mathscr{F}\rangle$ of a $\mathfrak{g}$-valued test shift $\langle\delta\alpha|$ with respect to the $\Lambda^3(M^3)$-valued coupling $\langle\,,\,\rangle$ refers to $\mathfrak{g}^*$ at the level of Lie algebras (i.e., regardless of the ghost parity and regardless of any tensor products with spaces of differential forms). This attributes the left-hand sides of Euler–Lagrange equations $\mathscr{E}_{\text{MC}}$ with $\mathfrak{g}^* \otimes \Lambda^2(M^3)$. However, we note that the pair of canonically conjugate variables would be α for $\mathfrak{g} \otimes \Lambda^1(M^3)$ and $\alpha^\dagger$ for $\Pi\mathfrak{g}^* \otimes \Lambda^2(M^3)$ whenever the Maurer–Cartan equations $\mathscr{E}_{\text{MC}}$ are brute-force labelled by using the respective unknowns, that is, if the metric tensor t_{ij} is not taken into account in the coupling $\langle\delta\alpha, \mathscr{F}\rangle$.

[l] We recall that the Koszul–Tate component of the full BV-differential D_{BV} is addressed in [38] by using the language of infinite jet bundles — whereas it is the BRST-component of D_{BV} which we focus on in this paper.

argument is supported by the following fact [17]: let $n \geqslant 3$ for M^n, suppose $\mathscr{E}$ is nonoverdetermined, and take a finite-dimensional Lie algebra $\mathfrak{g}$, then every $\mathfrak{g}$-valued zero-curvature representation α for $\mathscr{E}$ is gauge equivalent to zero (i.e., there exists $g \in C^\infty(\mathscr{E}^\infty, G)$ such that $\alpha = \mathrm{d}_h g \cdot g^{-1}$). It is remarkable that Marvan's homological technique, which contributed with the anchor ∂_α to our construction of non-Abelian variational Lie algebroids, was designed for effective inspection of the spectral parameters' (non)removability at $n = 2$ but *not* in the case of higher dimensions $n \geqslant 3$ of the base M^n.

We conclude that the approach to quantisation of kinematically integrable systems is not restricted by the BV-technique only; for one can choose between the former and, e.g., flat deformation of (structures in) equation (0.1) to the quantum setup of (0.2). It would be interesting to pursue this alternative in detail towards the construction of quantum groups [10] and approach of [32, 35] to quantum inverse scattering and quantum integrable systems. This will be the subject of another paper.

Discussion

Non-Abelian variational Lie algebroids which we associate with the geometry of $\mathfrak{g}$-valued zero-curvature representations are the simplest examples of such structures in a sense that the bracket $[\,,\,]_A$ on the anchor's domain is *a priori* defined in each case by the Lie algebra $\mathfrak{g}$. That linear bracket is independent of either base points $x \in M^n$ or physical fields $\phi(x)$. Another example of equal structural complexity is given by the gauge algebroids in Yang–Mills theory [3]. Indeed, the bracket $[\,,\,]_A$ on the anchor's domain of definition is then completely determined by the multiplication table of the structure group for the Yang–Mills field. The case of variational Poisson algebroids [14, 25] is structurally more complex: to determine the bi-differential bracket $[\,,\,]_A$ it suffices to know the anchor A; however, the bracket can explicitly depend on the (jets of) fields or on base points. The full generality of variational Lie algebroids setup is achieved for 2D Toda-like systems or gauge theories beyond Yang–Mills (e.g., for gravity). Therefore, the objects which we describe here mediate between the Yang–Mills and Chern–Simons models. It is remarkable that "reasonable" Chern–Simons models can in retrospect narrow the class of admissible base manifolds M^n for (gauge) field theories; for the quantum objects determine topological invariants of threefolds (e.g., via knot theory [36, 41]). Here we also admit that a triviality of the boundary conditions is assumed by default throughout this paper (see footnote d on p. 7 and also [2]). This is of course a model situation; a selection of "reasonable" geometries could in principle overload the setup with non-vanishing boundary terms.

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Appendix A. Lie algebroids: an overview

For consistency, let us recall the standard construction of a Lie algebroid over a usual smooth manifold N^m. By definition (see below) it is a vector bundle $\xi\colon \Omega^{d+m}\to N^m$ such that the $C^\infty(N)$-module $\Gamma(\xi)$ of its sections is endowed with a Lie algebra structure $[\,,\,]_A$ and with an anchor,

$$A\in \mathrm{Mor}(\xi,TN)\simeq \mathrm{Hom}_{C^\infty(N)}(\Gamma(\xi),\Gamma(TN)),\tag{A.1}$$

which satisfies Leibniz rule (A.5) for $[\,,\,]_A$. By introducing the odd neighbour $\Pi\xi\colon \Pi\Omega\to N$ of the vector bundle ξ, one represents [37] the Lie algebroid over N in terms of an odd-parity derivation Q in the ring $C^\infty(\Pi\Omega)\simeq\Gamma(\bigwedge^\bullet\Omega^*)$ of smooth functions on the total space $\Pi\Omega$ of the new superbundle $\Pi\xi$.

Let us indicate in advance the elements of the classical definition which are irreparably lost as soon as the base manifold becomes the total space of an infinite jet bundle $\pi_\infty\colon J^\infty(\pi)\to M^n$ for a given vector bundle π over the new base.[m] The new anchor almost always becomes a positive order operator in total derivatives; it takes values in the space of π_∞-vertical, evolutionary vector fields that preserve the Cartan distribution on $J^\infty(\pi)$. But Newton's binomial formula for the derivatives in A prescribes that the old identification $A(f\cdot\mathcal{X})=f\cdot A(\mathcal{X})$ of the two module structures for $\Gamma(\Omega)\ni\mathcal{X}$ and $\Gamma(TN)$ is no longer valid (and isomorphism (A.1) is lost). Simultaneously, Leibniz rule (A.5) is not valid, e.g., even if one takes $A=\mathrm{id}$ for $\xi=\pi$.

To resolve the arising obstructions, for the new definition of a *variational* Lie algebroid over $J^\infty(\pi)$ we take the proven Frobenius property,

$$[\mathrm{im}A,\mathrm{im}A]\subseteq\mathrm{im}A,\tag{A.2}$$

of the anchor to be the Lie algebra homomorphism $\big(\Gamma\Omega,[\,,\,]_A\big)\to\big(\Gamma(TN),[\,,\,]\big)$. In other words, we postulate an implication but not the initial hypothesis of classical construction. Such resolution

[m] To recognize the old manifold N^m in this picture and to understand where the new bundle π over M^n stems from, one could view N^m as a fibre in a locally trivial fibre bundle π over M^n, so that the new anchor takes values in $\Gamma(\pi_\infty^*(T\pi))$ for the bundle induced over $J^\infty(\pi)$ from the tangent $T\pi$ to π. It is then readily seen that the classical construction corresponds to the special case $n=0$ and $M^n=\{\mathrm{pt}\}$ (equivalently, one sets $\Gamma(\pi)\simeq N^m$ so that only constant section are allowed), see Fig. 3. However, in a generic situation of non-constant smooth sections one encounters differential operators $A\colon\Gamma(\pi_\infty^*(\xi))\to\Gamma(\pi_\infty^*(T\pi))$ for $\xi\colon\Omega^{n+d}\to M^n$; likewise, the 'functions' standing in coefficients of all object become differential functions of arbitrary finite order on $J^\infty(\pi)$.

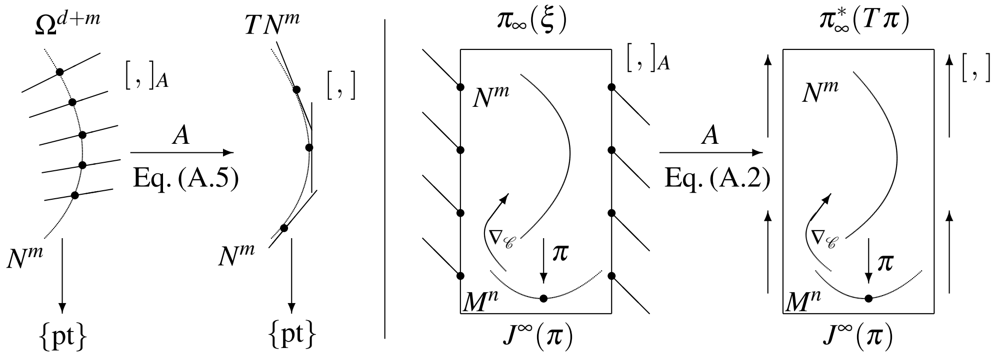

Fig. 3. From Lie algebroids $(\xi, A, [\,,]_A)$ to variational Lie algebroids $(\pi_\infty^*(\xi), A, [\,,]_A)$.

was proposed in [25] for the (graded-)commutative setup of Poisson geometry on $J^\infty(\pi)$ or for the geometry of 2D Toda-like systems and BV-formalism for gauge-invariant models such as the Yang-Mills equation (see [22] and also [3] in which an attempt to recognize the classical picture is made in a manifestly jet-bundle setup). In this paper we show that the new approach is equally well applicable in the non-Abelian case of Lie algebra-valued zero-curvature representations for partial differential equations $\mathscr{E}^\infty \subseteq J^\infty(\pi)$ (which could offer new insights in the arising gauge cohomology theories [33]).

A.1. *The classical construction of a Lie algebroid*

Let N^m be a smooth real m-dimensional manifold ($1 \leq m \leq +\infty$) and denote by $\mathscr{F} = C^\infty(N^m)$ the ring of smooth functions on it. The space $\varkappa = \Gamma(TN)$ of sections of the tangent bundle TN is an $\mathscr{F}$-module. Simultaneously, the space $\varkappa$ is endowed with the natural Lie algebra structure $[\,,]$ which is the commutator of vector fields,

$$[X,Y] = X \circ Y - Y \circ X, \qquad X,Y \in \Gamma(TN). \tag{A.3}$$

As usual, we regard the tangent bundle's sections as first order differential operators with zero free term.

The $\mathscr{F}$-module structure of the space $\Gamma(TN)$ manifests itself for the generators of $\varkappa$ through the Leibniz rule,

$$[fX,Y] = (fX) \circ Y - f \cdot Y \circ X - Y(f) \cdot X, \qquad f \in \mathscr{F}. \tag{A.4}$$

The coefficient $-Y(f)$ of the vector field X in the last term of (A.4) belongs again to the prescribed ring $\mathscr{F}$.

Let $\xi\colon \Omega^{m+d} \to N^m$ be another vector bundle over N and suppose that its fibres are d-dimensional. Again, the space $\Gamma\Omega$ of sections of the bundle ξ is a module over the ring $\mathscr{F}$ of smooth functions on the manifold N^m.

Definition ([37]). A *Lie algebroid* over a manifold N^m is a vector bundle $\xi\colon \Omega^{d+m} \to N^m$ whose space of sections $\Gamma\Omega$ is equipped with a Lie algebra structure $[\,,]_A$ together with a bundle morphism $A\colon \Omega \to TN$, called the *anchor*, such that the Leibniz rule

$$[f \cdot \mathcal{X}, \mathcal{Y}]_A = f \cdot [\mathcal{X}, \mathcal{Y}]_A - \big(A(\mathcal{Y})f\big) \cdot \mathcal{X} \tag{A.5}$$

holds for any $\mathcal{X}, \mathcal{Y} \in \Gamma\Omega$ and any $f \in C^\infty(N^m)$.

Example. Lie algebras are toy examples of Lie algebroids over a point. The other standard examples are the tangent bundle and the Poisson algebroid structure of the cotangent bundle to a Poisson manifold [31].

Lemma ([16])**.** *The anchor A maps the bracket $[\,,\,]_A$ for sections of the vector bundle ξ to the Lie bracket $[\,,\,]$ for sections of the tangent bundle to the manifold N^m.*

This property is a consequence of Leibniz rule (A.5) and the Jacobi identity for the Lie algebra structure $[\,,\,]_A$ in $\Gamma\Omega$. Remarkably, the assertion of this Lemma is often *postulated* (for convenience, rather than derived) as a part of the definition of a Lie algebroid, e. g., see [37, 39] vs [16, 31].

In the course of transition from usual manifolds N^m to jet spaces $J^\infty(\pi)$ it is natural that maps of spaces of sections become nonnegative-order linear differential operators. For example, the anchors will be operators in total derivatives $A \in \mathscr{C}\mathrm{Diff}\big(\Gamma(\pi_\infty^*(\xi)) \to \Gamma(\pi_\infty^*(T\pi))\big)$ for spaces of sections of induced vector bundles; note that the π_∞-vertical component of the tangent bundle to $J^\infty(\pi)$ is the target space.[n] Whenever that differential order is strictly positive, one loses the property of A to be a homomorphism over the algebra $\mathscr{F}(\pi) = C^\infty(J^\infty(\pi))$ of differential functions of arbitrary finite order. Indeed, consider the first-order anchor $\partial_\alpha = [\cdot,\alpha] + \mathrm{d}_h$, which we discuss in this paper (cf. [33]): even though $[f\cdot p,\alpha] = f\cdot[p,\alpha]$, the horizontal differential d_h acts by the Leibniz rule so that $\partial_\alpha(f\cdot p) \neq f\cdot\partial_\alpha(p)$ if $f \neq \mathrm{const}$. We see that such map of horizontal module of sections for a bundle $\pi_\infty^*(\xi)$ induced over $J^\infty(\pi)$ is not completely determined by the images of a basis of local sections in ξ, which is in contrast with the classical case in (A.1).

Likewise, the Leibniz rule expressed by (A.5) does not hold whenever a section $\mathcal{Y} \in \Gamma(\pi_\infty^*(\xi)) \simeq \Gamma(\xi) \otimes_{C^\infty(M)} C^\infty(J^\infty(\pi))$ contains derivatives u_σ of fibre coordinates u in π. A (counter)example is as follows: take $\xi = T\pi$ and set $A = \mathrm{id}\colon \big(\Gamma(\pi_\infty^*(T\pi)), [\,,\,]_A\big) \to \big(\Gamma(\pi_\infty^*(T\pi)), [\,,\,]\big)$, where both Lie algebra structures are the commutator of evolutionary vector fields. Let $\mathcal{X}, \mathcal{Y} \in \Gamma\big(\pi_\infty^*(T\pi)\big)$ and $f \in C^\infty(J^\infty(\pi))$. Then we have that

$$[f\mathcal{X},\mathcal{Y}]_A = \partial_{f\mathcal{X}}^{(u)}(\mathcal{Y}) - \partial_{\mathcal{Y}}^{(u)}(f\cdot\mathcal{X}) = f\cdot[\mathcal{X},\mathcal{Y}]_A - A(\mathcal{Y})(f)\cdot\mathcal{X} + \sum_{|\sigma|>0}\sum_{\substack{\rho\cup\tau=\sigma\\|\rho|>0}} \frac{\mathrm{d}^{|\rho|}}{\mathrm{d}x^\rho}(f)\cdot\frac{\mathrm{d}^{|\tau|}}{\mathrm{d}x^\tau}(\mathcal{X})\cdot\frac{\partial}{\partial u_\sigma}(\mathcal{Y}).$$

As soon as the above two ingredients of the classical definition are lost, we take for definition of an anchor in a *variational* Lie algebroid over $J^\infty(\pi)$ the involutivity $[\mathrm{im}A, \mathrm{im}A] \subseteq \mathrm{im}A$ of image of a linear operator $A \in \mathscr{C}\mathrm{Diff}\big(\Gamma(\pi_\infty^*(\xi)), \Gamma(\pi_\infty^*(T\pi))\big)$ whose values belong to the space of generating sections of evolutionary vector fields on $J^\infty(\pi)$ (alternatively, the anchor could take values in a smaller Lie algebra of infinitesimal symmetries for a given equation $\mathscr{E}^\infty \subseteq J^\infty(\pi)$). Notice that the anchor is then a Lie algebra homomorphism by construction, namely, $A\colon \big(\Gamma(\pi_\infty^*(\xi)), [\,,\,]_A\big) \to \big(\varkappa(\pi), [\,,\,]\big)$. Note further that, on one hand, the bracket $[\,,\,]_A$ could be induced on $\Gamma(\pi_\infty^*(\xi))/\ker A$ by the property $[A(p_1), A(p_2)] = A([p_1,p_2]_A)$ of commutation closure for the image of A. (Such is the geometry of Liouville-type Toda-like systems or the BRST- and BV-approach to gauge field models, see [22, 24, 25] and references therein). On the other hand, the bracket $[\,,\,]_A$ can be present *ab initio* in the picture: such is the case of Hamiltonian operators A in the Poisson formalism or the

[n] We recall that both junior and senior Hamiltonian differential operators have *positive* differential orders for all Drinfel'd–Sokolov hierarchies associated with the root systems; this construction yields a class of variational Poisson algebroids. The anchors which are linear operators of zero differential order are a rare exception (however, see [23] in this context).

geometry of zero-curvature representations (indeed, we then have $[\,,\,]_A = [\,,\,]_{\mathfrak{g}}$ for the Lie algebra $\mathfrak{g}$ of a gauge group G). This alternative yields four natural examples of variational Lie algebroids.

A.2. *The odd neighbour* $\Pi\xi\colon \Pi\Omega \to N^m$ *and differential* $Q^2 = 0$.

The odd neighbour of a vector bundle $\xi\colon \Omega^{m+d} \to N^m$ over a smooth real manifold N^m is the vector bundle $\Pi\xi\colon \Pi\Omega^{m+d} \to N^m$ over the same base and with the same vector space $\mathbb{R}^d$ taken as the prototype for the fibre over each point $x \in \mathscr{U}_\alpha \subseteq N^m$: the coordinate diffeomorphism is $\varphi_\alpha\colon \mathscr{U}_\alpha \times \mathbb{R}^d \to \Pi\Omega^{d+m}$. Moreover, the topology of the bundle $\Pi\xi$ coincides with that of ξ so that the gluing transformations $g^{\Pi}_{\alpha\beta} \in GL(d,\mathbb{R})$ in the fibres over intersections $\mathscr{U}_\alpha \cap \mathscr{U}_\beta \subseteq N^m$ of charts, smoothly depending on $x \in \mathscr{U}_\alpha \cap \mathscr{U}_\beta$, are exactly the same as the fibres' reparametrizations $g_{\alpha\beta}(x)$ in the bundle ξ. However, notice that these *linear* mapping can not feel any grading of the object which they transform (in particular, $g_{\alpha\beta}$ can not grasp the $\mathbb{Z}_2$-valued parity of such $\mathbb{R}^d$); this indifference is the key element in a construction of the odd neighbour. Namely, let the coordinates $b^1,\ldots,b^d$ along the fibres $(\Pi\xi)^{-1}(x) \simeq \mathbb{R}^d$ be $\mathbb{Z}_2$-parity odd,[o] i.e., introduce the $\mathbb{Z}_2$-grading $|\cdot|\colon x^i \mapsto \bar{0},\ b^j \mapsto \bar{1}$ for the ring of smooth $\mathbb{R}$-valued functions on the total space $\Pi\Omega$ of the superbundle (the grading then acts by a multiplicative group homomorphism $|\cdot|\colon C^\infty(\Pi\Omega) \to \mathbb{Z}_2$). We have that $C^\infty(\Pi\Omega) \simeq \Gamma(\bigwedge^\bullet \Omega^*)$, where Ω^* denotes the space of fibrewise-linear functions on Ω. By construction, the new space of graded coordinate functions on $\Pi\Omega$ is an $\mathbb{R}$-algebra and a $C^\infty(N)$-module.

Notice further that the space of the bundle's sections in principle stays intact; however, it is not the sections of $\Pi\xi$ which will be explicitly dealt with in what follows but it is a convenient handling of cochains and cochain maps for $\Gamma(\xi)$ by coding those objects and structures in terms of fibrewise-homogeneous functions on $\Pi\Omega$.

Remark. It is important to distinguish between sections $p \in \Gamma(\xi)$, $p\colon N^m \to \Omega^{m+d}$, and fibre coordinates p^j on the total space Ω of the vector bundle ξ. Indeed, $\partial p^j/\partial x^i \equiv 0$ by definition whereas the value at $x \in N^m$ of a derivative $\frac{\partial}{\partial x^i}(p^j)(x)$ of a section p could be any number. In particular, consider the Jacobi identity for the Lie algebra structure $[\,,\,]_A\colon \Gamma(\xi)\times\Gamma(\xi) \to \Gamma(\xi)$ in a Lie algebroid. Let $p_\mu = p^i_\mu \vec{e}_i$ be sections of ξ, here $\mu = 1, 2, 3$, and denote by $c^k_{ij}(x)$ the values at $x \in N^m$ of the structure constants of $[\,,\,]_A$ with respect to a natural basis $\vec{e}_i$ of local sections. Then we have that

$$
\begin{aligned}
0 = \sum_{\circlearrowleft}[[p_1,p_2]_A,p_3]_A &= \sum_{\circlearrowleft}[p^i_1 c^k_{ij}(x) p^j_2\cdot\vec{e}_k, p^\ell_3\cdot\vec{e}_\ell]_A \\
&= \sum_{\circlearrowleft} p^i_1 p^j_2 p^\ell_3 \cdot \Big\{ c^k_{ij}(x) c^n_{kl}(x)\cdot\vec{e}_n - \big(A\big|_x(\vec{e}_\ell)\big)(c^k_{ij}(x))\cdot\vec{e}_k \Big\} \\
+\sum_{\circlearrowleft} c^k_{ij}(x)\cdot\Big\{ p^i_1 p^j_2 \cdot \big(A\big|_x(\vec{e}_k)\big)(p^\ell_3)(x)\cdot\vec{e}_\ell &- p^j_2 p^\ell_3\cdot\big(A\big|_x(\vec{e}_\ell)\big)(p^i_1)(x)\cdot\vec{e}_k \\
&- p^i_1 p^\ell_3\cdot\big(A\big|_x(\vec{e}_\ell)\big)(p^j_2)(x)\cdot\vec{e}_k\Big\}. \qquad \text{(A.6)}
\end{aligned}
$$

Clearly, if the coefficients p^i_μ are viewed as local coordinates along fibres in Ω over $x \in N^m$ parametrized by $x^1,\ldots,x^m$, then the vector fields $A(\vec{e}_\ell) \in \Gamma(TN)$ no longer act on such p^i_μ's so that the entire last sum in (A.6) vanishes.

[o] The parity reversion $\Pi\colon p \leftrightarrows b$ acts on the fibre coordinates but not on a basis $\vec{e}_i$ in $\mathbb{R}^d$. To keep track of a distinction between the two geometries, we formally denote by $\mathfrak{e}_i = \Pi\vec{e}_i$ the basis in $\mathbb{R}^d$ which referes to the $\mathbb{Z}_2$-graded setup.

We refer to [22, 25] for a discussion on the immanent presence and recovery of the 'standard,' vanishing terms in the course of transition $C^\infty(\Pi\Omega) \to C^\infty(\Omega) \to \mathrm{Alt}\big(\Gamma(\xi)\times\cdots\times\Gamma(\xi)\to\Gamma(\xi)\big)$ from homogeneous functions of the odd fibre coordinates to $\Gamma(\xi)$-valued cochains (and cochain maps such as the Lie algebroid differential d_A). A detailed analysis of properties and interrelations between the four neighbours $\mathfrak{g}$, $\Pi\mathfrak{g}$, $\mathfrak{g}^*$, and $\Pi\mathfrak{g}^*$ is performed in [39] (here $m = 0$, $N^m = \{\mathrm{pt}\}$, and the Lie algebroid Ω is a Lie algebra $\mathfrak{g}$).

Proposition ([37]). *The Lie algebroid structure on Ω is encoded by the homological vector field Q on $\Pi\Omega$, i.e., by a derivation in the ring $C^\infty(\Pi\Omega) = \Gamma(\bigwedge^\bullet \Omega^*)$,*

$$Q = A_i^\alpha(x)\, b^i \frac{\partial}{\partial x^\alpha} - \tfrac{1}{2} b^i c_{ij}^k(x)\, b^j \frac{\partial}{\partial b^k}, \qquad [Q,Q] = 0 \quad\Longleftrightarrow\quad 2Q^2 = 0,$$

where

- *(x^α) is a system of local coordinates near a point $x \in N^m$,*
- *(p^i) are local coordinates along the d-dimensional fibres of Ω and (b^i) are the respective coordinates on $\Pi\Omega$, and*
- *the formula $[\vec{e}_i, \vec{e}_j]_A = c_{ij}^k(x)\,\vec{e}_k$ gives the structure constants for a d-element local basis $(\vec{e}_i)$ of sections in $\Gamma\Omega$ over the point x, and $A(\vec{e}_i) = A_i^\alpha(x)\cdot\partial/\partial x^\alpha$ is the image of $\vec{e}_i$ under the anchor A.*

Sketch of the proof. The reasoning goes in parallel with the proof of Theorem 2.1. First, we recall that the anchor $A = \|A_i^\alpha\|_{1\leqslant i\leqslant d}^{1\leqslant\alpha\leqslant m}$ is the Lie algebra homomorphism by the previous Lemma on p. 23. Second, we note that the homogeneous (in odd-parity coordinates b^j) coefficients of $\partial/\partial b^k$, $1 \leqslant k \leqslant d$, in Q^2 encode the tri-linear, totally skew-symmetric map $\omega_3\colon \Gamma(\xi)\times\Gamma(\xi)\times\Gamma(\xi)\to\Gamma(\xi)$ whose value at any p_1, p_2, $p_3 \in \Gamma(\xi)$ is twice the right-hand side of Jacobi's identity (A.6). Here we use the fact that cyclic permutations of three objects are even (in terms of permutation's $\mathbb{Z}_2$-parity), whence it is legitimate to extend the summation $\sum_{\circlearrowleft}$ to a sum over the entire permutation group S_3:

$$\sum_{\circlearrowleft} \omega_3\big(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}\big) = \frac{1}{2}\sum_{\sigma\in S_3} (-)^\sigma\, \omega_3\big(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}\big).$$

The presence of zero section in the left-hand side of Jacobi identity (A.6) implies that the respective coefficient of $\partial/\partial b$ in Q vanishes.[p] □

Remark. The coefficient $+\frac{1}{2}$ in the homological evolutionary vector field Q in Theorem 2.1, but not the opposite value $-\frac{1}{2}$ in the canonical formula (see the Proposition on p. 25) is due to our choice of sign in a notation for the zero-curvature representation $\alpha = A_i \cdot \mathrm{d}x^i$: one sets either $\Psi_{x^i} + A_i\Psi = 0$ or $\Psi_{x^i} = A_i\Psi$ for the wave function Ψ. The second option is adopted by repetition but it tells us that the gauge connection's $\mathfrak{g}$-valued one-form is minus α.

Remark. The correspondence $f_k \leftrightarrow \omega_k$ between homogeneous functions $f_k(x;b,\ldots,b) \in C^\infty(\Pi\Omega)$ on the total space of the superbundle $\Pi\xi$ and k-chain maps $\omega_k\colon \Gamma(\xi)\times\cdots\times\Gamma(\xi)\to C^\infty(N)$ correlates the homological vector field Q with the Lie algebroid differential d_A that acts by the standard

[p]Notice that the second step of this reasoning is simplified further in the case of non-Abelian variational Lie algebroids (see p. 8) because in that case the bracket $[\,,\,]_A$ is a given Lie algebra structure in $\mathfrak{g}$; it is described globally by using the structure constants c_{ij}^k regardless of the base manifold.

Cartan formula. Namely, the following diagram is commutative,

$$\begin{array}{ccc} \mathrm{d}_A\colon\ \omega_k & \longrightarrow & \omega_{k+1} \\ \Big\downarrow & & \Big\downarrow \\ Q\colon\ f_k & \longrightarrow & f_{k+1}. \end{array}$$

The wedge product of k- and ℓ-chains corresponds under the vertical arrows of this diagram to the ordinary $\mathbb{Z}_2$-graded multiplication of the respective functions from $C^\infty(\Pi\Omega)$.

The main examples of this construction are the de Rham differential on a manifold N^m (as before, set $\xi := \pi$ and let $A = \mathrm{id}$), the Chevalley–Eilenberg differential for a Lie algebra $\mathfrak{g}$ (let $m = 0$, $N^m = \{\mathrm{pt}\}$, and take $(\Omega, [\,,\,]_A) = (\mathfrak{g}, [\,,\,]_{\mathfrak{g}})$ and $A = 0$), and the de Rham differential on a symplectic manifold (here $\xi\colon \Lambda^1(N^m) \to N^m$, $A = [\![P,\cdot]\!]$ is the Poisson differential given by a bivector P satisfying $[\![P,P]\!] = 0$ and having the inverse symplectic two-form P^{-1}, and $[\,,\,]_A$ is the Koszul–Dorfman–Daletsky–Karasëv bracket [9, 31]).

The Hamiltonian homological evolutionary vector field Q that encodes the *variational* Poisson algebroid structure over a jet space $J^\infty(\pi)$ was de facto written in [14]. The BRST-differential Q is another example of such construction over jet spaces $J^\infty(\pi) \supseteq \mathscr{E}^\infty$ containing the Euler–Lagrange equations for gauge-invariant models.