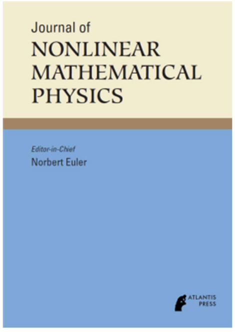

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The Ablowitz–Ladik hierarchy integrability analysis revisited: the vertex operator solution representation structure

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A regular gradient-holonomic approach to studying the Lax type integrability of the Ablowitz–Ladik hierarchy of nonlinear Lax type integrable discrete dynamical systems in the vertex operator representation is presented. The relationship to the Lie-algebraic integrability scheme is analyzed and the connection with the τ-function representation is discussed.

Keywords: the Ablowitz–Ladik hierarchy; discrete Lax type integrability; Lie-algebraic approach; vertex operator structure

2010 Mathematics Subject Classification: 37K10, 17B69, 17B80

1. Introduction

In the recent years the discrete dynamical systems have been profoundly studied by many researches. Unlike continuous systems, the nonlinear lattice models are more complicated and harder to investigate due to non-uniqueness of discretization and nonlocal forms of discrete systems. When continuous system possesses some features belonging to its integrability such as a Lax pair as well as independent conserved laws and symmetries, then these characteristics should be kept under the discretization algorithm so that discretized system remains integrable. As the integrable systems are related to some linear isospectral problems, generating the corresponding integrable hierarchies, their unique description proved to be very effective [8–11, 19] by means of a so called τ-function, solving both the corresponding eigenfunction problems and presenting the special solutions to the dynamical systems under regard.

The τ-function, apparently having been already known [2] to Darboux, can be mainly viewed as a functional object generating the whole hierarchy of integrable dynamical systems. The τ-functions play the central role [3–7] in establishing the connections between integrable systems and quantum field theory, statistical mechanics or the theory of random matrices. They in general can be represented as determinants of infinite matrices or can be identified with the Fredholm determinant of the corresponding integral Gel'fand-Levitan-Marchenko equation used for exact solving the model under consideration. Within the context of the Zakharov and Shabat dressing method [23] the τ-function was also interpreted as the Fredholm determinant in [12]. As in many cases the corresponding τ-functional representations were obtained by means of both very complicated [3, 11, 19] and algorithmically fuzzy calculations, it was natural to reanalyze them applying a unique approach based on the analytical properties of the realated hierarchy of integrable dynamical systems. In the

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case of the AKNS-integrable hierarchy [13] the corresponding approach for finding the related τ-functional representation was devised in [14], drawing upon the generating vector field flow and its related vertex operator representation.

The present work is devoted to developing the approach of [14] in the case of the integrable Discrete Nonlinear Schrödinger evolution equation and the correspoding Ablowitz-Ladik integrable hierarchy which, as well known [1], is generated by the following linear discrete Lax type spectral problem

$$f_{n+1} = l_n[u,v;\lambda]f_n, \quad l_n[u,v;\lambda] := \begin{pmatrix} \lambda & u_n \\ v_n & \lambda^{-1} \end{pmatrix}, \tag{1.1}$$

where $f_n \in l_1(\mathbb{Z};\mathbb{C}^2)$, $l_n[u,v;\lambda] \in Aut\ \mathbb{C}^2$, $n \in \mathbb{Z}$, $\lambda \in \mathbb{C}$ is a spectral parameter, $(u,v) \in M \subset (\mathbb{C}^2)^{\mathbb{Z}/N\mathbb{Z}}$ for some $N \in \mathbb{Z}_+$, under the isospectral condition $d\lambda/dt = 0$, $t \in \mathbb{C}$. The latter gives rise to the infinite Ablowitz–Ladik hierarchy of integrable differential-difference dynamical systems on the manifold M. The first of them being the flows

$$\begin{aligned} \frac{d}{dt_0}\begin{pmatrix} u_n \\ v_n \end{pmatrix} &= \begin{pmatrix} u_n \\ -v_n \end{pmatrix}, \\ \frac{d}{dt_1}\begin{pmatrix} u_n \\ v_n \end{pmatrix} &= \begin{pmatrix} u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}) \\ -v_{n+1} + 2v_n - v_{n-1} - v_n u_n (v_{n+1} + v_{n-1}) \end{pmatrix}, \end{aligned} \tag{1.2}$$

are known as the scaling and the Discrete Nonlinear Schrödinger evolution equations and have many interesting applications [29] in modern plasma physics.

To obtain the vertex representation of the Ablowitz–Ladik hierarchy, we will make use of a gradient-holonomic approach devised in [15, 24, 28] and apply it to the spectral problem (1.1). We will also show that the obtained vertex operator representation, expressed by means of the τ-function, coincides, in particular, with the one obtained in [17, 18] using a different, vague and very cumbersome approach.

To begin with, let us construct a matrix solution to the linear equation

$$\tilde{F}_{m+1,n}(\lambda) = l_m[u,v;\lambda]\tilde{F}_{m,n}(\lambda) \tag{1.3}$$

under the initial condition

$$\tilde{F}_{m,n}(\lambda)|_{m=n} = \mathbf{I} + O(1/\lambda) \tag{1.4}$$

as $\lambda \to \infty$ for all $n \in \mathbb{Z}$, where $\mathbf{I}$ is the identity map. Herein a matrix solution to the linear equation

$$\tilde{F}_{m+1,n}(z) = l_m[u,v;\lambda]\tilde{F}_{m,n}(z) \tag{1.5}$$

under the initial condition

$$\tilde{F}_{m,n}(z)|_{m=n} = \mathbf{I} + O(1/z) \tag{1.6}$$

as $\lambda := 1/z \to 0$ for all $n \in \mathbb{Z}$, similarly to the discrete linear Lax type problems (1.3) and (1.5), is singular as $\lambda \to \infty$ and as $\lambda = 1/z \to 0$. As a result of the calculations, we find that the matrix

$$\tilde{F}_{m,n}(\lambda) = \begin{pmatrix} \tilde{e}^{(1)}_{m,n}(\lambda) & -\lambda^{-1}\tilde{u}_m(\lambda)\tilde{e}^{(2)}_{m,n}(\lambda) \\ \lambda^{-1}\tilde{v}_{m-1}(\lambda)\tilde{e}^{(1)}_{m,n}(\lambda) & \tilde{e}^{(2)}_{m,n}(\lambda) \end{pmatrix}, \tag{1.7}$$

where

$$\tilde{e}^{(1)}_{m,n}(\lambda) := \prod_{k=n}^{m-1}(\lambda + \lambda^{-1}u_k\tilde{v}_{k-1}(\lambda)), \tilde{e}^{(2)}_{n,m}(\lambda) := \prod_{k=n}^{m-1}\lambda^{-1}(1 - v_k\tilde{u}_k(\lambda)), \tag{1.8}$$

together with functional relationships

$$\begin{pmatrix} \tilde{u}_n(\lambda) \\ \tilde{v}_n(\lambda) \end{pmatrix} = \begin{pmatrix} u_n + \lambda^{-2}\tilde{u}_{n+1}(\lambda)[1 - \tilde{u}_n(\lambda)v_n] \\ v_n + \lambda^{-2}\tilde{v}_{n-1}(\lambda)[1 - u_n\tilde{v}_n(\lambda)] \end{pmatrix} \tag{1.9}$$

solves the problem (1.3), (1.4) as $\lambda \to \infty$, and the matrix

$$\tilde{F}_{m,n}(z) = \begin{pmatrix} \tilde{e}^{(1)}_{m,n}(z) & -z^{-1}\tilde{u}_m(z)\tilde{e}^{(2)}_{m,n}(z\lambda) \\ z^{-1}\tilde{v}_m(\lambda)\tilde{e}^{(1)}_{m,n}(z) & \tilde{e}^{(2)}_{m,n}(z) \end{pmatrix}, \tag{1.10}$$

where

$$\tilde{e}^{(1)}_{m,n}(z) := \prod_{k=n}^{m-1}(z^{-1} - z^{-1}u_k\tilde{v}_k(z)), \tilde{e}^{(2)}_{n,m}(\lambda) := \prod_{k=n}^{m-1}(z + z^{-1}v_k\tilde{u}_{k-1}(z)) \tag{1.11}$$

together with functional relationships

$$\begin{pmatrix} \tilde{u}_n(z) \\ \tilde{v}_n(z) \end{pmatrix} = \begin{pmatrix} u_n + z^{-2}\tilde{u}_{n-1}(z)[1 - \tilde{u}_n(z)v_n] \\ v_n + z^{-2}\tilde{v}_{n+1}(\lambda)[1 - u_n\tilde{v}_n(z)] \end{pmatrix} \tag{1.12}$$

solves the problem (1.3), (1.4) as $\lambda = 1/z \to 0$. It is worth mentioning that relationships (1.9) and (1.12) imply the following important limit properties:

$$\lim_{\lambda\to\infty}\tilde{u}_n(\lambda) = u_n,\ \lim_{\lambda\to\infty}\tilde{v}_n(\lambda) = v_n,\ \lim_{z\to\infty}\tilde{u}_n(z) = u_n,\ \lim_{\lambda\to\infty}\tilde{v}_n(z) = v_n, \tag{1.13}$$

holding for all $n \in \mathbb{Z}$.

Now, on the basis of the matrix expressions (1.7) and (1.10), one can easily construct the fundamental matrices

$$F_{m,n}(\lambda) := \tilde{F}_{m,n}(\lambda)\tilde{F}^{-1}_{n,n}(\lambda) \tag{1.14}$$

and

$$F_{m,n}(z) := \tilde{F}_{m,n}(z)\tilde{F}^{-1}_{n,n}(z), \tag{1.15}$$

respectively solving the linear problems (1.3) and (1.5) under the initial conditions

$$F_{m,n}(\lambda)|_{m=n} = \mathbf{I}+O(1/\lambda),\ F_{m,n}(z)|_{m=n} = \mathbf{I}+O(1/z) \tag{1.16}$$

for all $n \in \mathbb{Z}$ as $\lambda, z \to \infty$.

Now taking into account that the manifold M is N-periodic, one can construct the monodromy matrices

$$S_{n,-}(\lambda) := F_{n+N,n}(\lambda) \tag{1.17}$$

as $\lambda \to \infty$, and

$$S_{n,+}(z) := F_{n+N,n}(z) \tag{1.18}$$

as $\lambda = 1/z \to 0$, for any $n \in \mathbb{Z}$. The monodromy matrices (1.17) and (1.18) satisfy the useful properties

$$S_{n+N,-}(\lambda) := S_{n,-}(\lambda), \;\; S_{n+N,+}(z) := S_{n,+}(z), \det S_{n,-}(\lambda) = 1 = \det S_{n,+}(z), \tag{1.19}$$

holding for any $n \in \mathbb{Z}$ and $\lambda, z \to \infty$. Keeping in mind the importance of invariants and Poissonian structures related with the linear spectral problems (1.1) or (3.36), the next section is devoted to the study of their basic Lie-algebraic properties and connections with the so called vertex operator representation of the whole Ablowitz–Ladik hierarchy of integrable differential-difference dynamical systems on the manifold M.

2. Differential-difference Lax integrable dynamical systems and their Lie-algebraic structure

We will consider the Lie-algebraic aspects of differential-difference dynamical systems associated with the generalized Lax linear difference spectral problem:

$$f_{n+1} = l_n[u,v;\lambda] f_n, \tag{2.1}$$

where $f \in l_\infty(\mathbb{Z};\mathbb{C}^2)$, $\;l_n := l_n[u,v;\lambda] \in G := GL_2(\mathbb{C}) \otimes \mathbb{C}(\lambda, \lambda^{-1})$ for $n \in \mathbb{Z}_N$ with $N \in \mathbb{Z}_+, \lambda \in \mathbb{C}$, is a spectral parameter and $(u,v)^\intercal \in M \subset (\mathbb{C}^2)^{\mathbb{Z}_N}$ is an a priori given discrete N-periodic finite-dimensional manifold. To describe the Lie-algebraic structure of the Lax integrable dynamical systems related with the spectral problem (2.1), let us first define the matrix product-group $G^N := \bigotimes_{j=1}^{N} G$ and its action on the phase space $M_G^{(N)} := \{l_n \in G : n \in \mathbb{Z}_N\}$

$$G^N \times M_G^{(N)} \to M_G^{(N)}, \tag{2.2}$$

given as

$$\{g_n \in G : n \in \mathbb{Z}_N\} \times \{l_n \in G : n \in \mathbb{Z}_N\} = \{g_n^{-1} l_n g_{n-1} \in G : n \in \mathbb{Z}_N\}. \tag{2.3}$$

Concerning the action (2.3), a functional $\gamma \in \mathscr{D}(M_G^{(N)})$ is invariant iff the following discrete relationship

$$\operatorname{grad}\gamma(l_n) l_n = l_{n+1} \operatorname{grad}\gamma(l_{n+1}) \tag{2.4}$$

holds for all $n \in \mathbb{Z}_N$.

Concerning the Lie-algebraic analysis of (2.4), we will assume that the group G^N is identified with its co-adjoint space $T^*(G^N)$, which is locally isomorphic to the co-adjoint algebra $\mathscr{G}_N^*$, where $\mathscr{G}_N$ is the corresponding Lie algebra of the group Lie G^N, which is isomorphic, by definition, to the tangent space $T_e(G^N)$ at the group identity $e \in G^N$. In particular, any element $l \in G^N$ is injectively associated with the corresponding element $l \in T^*(G^N)$ acting on $T^*(G^N)$ as follows:

$$(l,X) := tr(lX), \tag{2.5}$$

where $l \in G^N$, $X \in T_e(G^N)$ and $tr : G^N \to \mathbb{C}$ is the trace operation on the group G^N: $trA := res_{\lambda=\infty} \sum_{j\in\mathbb{Z}} SpA_j[u,v;\lambda]$ for any $A \in G^N$. Owing to (2.5), one can introduce basic elements $e_l^* \in \mathscr{G}_N^*$ such that $le_l^* \in T_e^*(G^N)$ for any $l \in G^N \simeq T_e^*(G^N)$, where $e_l^* \in T_e^*(G^N)$ is defined via the relationship $e_l^*(Xl) := (l,X)$ for any $X \in T_e(G^N)$. Thus, we can define the set

$$\{\Phi_n = \mathrm{grad}\gamma(l_n)l_n : n \in \mathbb{Z}_N\} \tag{2.6}$$

belonging to the Lie algebra $\mathscr{G}_N \simeq T_e(G^N)$ and satisfying the following invariance property

$$\Phi_{n+1} = l_n\Phi_n(\lambda)l_n^{-1} \tag{2.7}$$

for any $n \in \mathbb{Z}_N$. The relationship (2.7) allows to define the function $\varphi : G \to \mathbb{C}$, which is invariant with respect to the adjoint action

$$G \times G \ni (g, S_n(\lambda)) \to g^{-1}S_n(\lambda)g \in G \tag{2.8}$$

for any $n \in \mathbb{Z}_N$ and such that

$$\gamma(l) = \varphi[S_N(\lambda)], \Phi_N = \mathrm{grad}\varphi[S_N(\lambda)]S_N(\lambda), \tag{2.9}$$

where, by definition, the expression

$$S_N(\lambda) = \prod_{j=1}^{\overleftarrow{N}} l_j[u,v;\lambda] \tag{2.10}$$

coincides with the proper monodromy matrix for the linear "spectral" problem (2.1). In virtue of (2.7), the matrices $\Phi_n = \mathrm{grad}\varphi[S_n(\lambda)]S_n(\lambda) \in \mathscr{G}$, $n \in \mathbb{Z}_N$, can be reconstructed from (2.10) and we can define the Poissonian flow on the matrices $S_n(\lambda) \in T_{S_n}^*(G), n \in \mathbb{Z}_N$, with respect to any $\mathscr{R}$-structure $\mathscr{R} : \mathscr{G} \to \mathscr{G}$ as

$$dS_n(\lambda)/dt = [\mathscr{R}(\mathrm{grad}\varphi[S_n(\lambda)]S_n(\lambda)), S_n(\lambda)] \tag{2.11}$$

for the invariant Casimir function $\varphi \in I(\mathscr{G}^*)$. In particular, this means

$$[\mathrm{grad}\varphi(S_n), S_n] = 0 \tag{2.12}$$

for all $n \in \mathbb{Z}_N$.

Remark. We need to mention here that the element $l \in T_e^*(G^N)$ is considered as one that is pulled back to the space $T_e^*(G^N)$ via the above mapping; namely,

$$G^N \ni l \to le_l^* \in T_e^*(G^N). \tag{2.13}$$

Taking into account (2.9), one can rewrite (2.11) in the equivalent form

$$dS_n/dt = [\mathscr{R}(\mathrm{grad}\gamma(l_n)l_n), S_n], \tag{2.14}$$

holding for all $n \in \mathbb{Z}_N$. This together with (2.7) makes it possible to retrieve [14, 16] the related evolution of elements $l_n \in G$, $n \in \mathbb{Z}_N$ in the form

$$\begin{aligned} dl_n/dt &= p_{n+1}(l)l_n - l_n p_n(l), \\ p_n(l) &:= \mathscr{R}(\mathrm{grad}\gamma(l_n)l_n), \end{aligned} \tag{2.15}$$

following from the relationships

$$\begin{aligned} S_n(\lambda) &= \psi_n(l)S_N(\lambda)\psi_n^{-1}(l), \\ \psi_n(l) &= \overleftarrow{\prod_{j=1}^{n}} l_j[u,v;\lambda]. \end{aligned} \tag{2.16}$$

The solution $f \in l_\infty(\mathbb{Z}, \mathbb{C}^P)$, with regard to the linear spectral problem (2.1), satisfies the associated temporal evolution equation

$$df_n/dt = p_n(l)f_n \tag{2.17}$$

for any $n \in \mathbb{Z}$. It is easy to check that the compatibility condition for the linear equations (2.1) and (2.17) is equivalent to the discrete Lax representation (2.15), which, upon reducing it on the group manifold M_G, gives rise to a nonlinear Lax integrable dynamical system on the discrete manifold M. This follows from the fact that all Casimir invariant functions, when reduced on the manifold M_G, are in involution with respect to the Poisson bracket

$$\{\tilde{\gamma}, \tilde{\xi}\}_{\mathscr{R}} := (l, [\mathrm{grad}\tilde{\gamma}(l), \mathscr{R}(\mathrm{grad}\tilde{\xi}(l)l] + [\mathscr{R}(l\mathrm{grad}\tilde{\gamma}(l)), \mathrm{grad}\tilde{\xi}(l)]) \tag{2.18}$$

on M, where $\tilde{\gamma}, \tilde{\xi} \in \mathscr{D}(\mathscr{G}_N^*)$ are arbitrary smooth functionals on $\mathscr{G}_N^* \simeq G^N$.

In the case when the $\mathscr{R}$-structure $\mathscr{R} = 1/2(P_+ - P_-)$, where $P_\pm : \mathscr{G} \to \mathscr{G}_\pm \subset \mathscr{G}$ are the projectors on the λ-positive and λ-negative degree subalgebras of the Lie algebra $\mathscr{G}$, the determining Lax type equation (2.15) generates two different flows: the negative one

$$\frac{d}{dt_{j,-}} l_n[u,v;\lambda] = (\lambda^{j+1}\tilde{S}_{n+1,-}(\lambda))_+ l_n[u,v;\lambda] - l_n[u,v;\lambda](\lambda^{j+1}\tilde{S}_{n,-}(\lambda))_+; \tag{2.19}$$

and the positive one

$$\frac{d}{dt_{j,+}} l_n[u,v;1/z] = (z^{j+1}\tilde{S}_{n+1,-}(z))_+ l_n[u,v;1/z] - l_n[u,v;1/z](z^{j+1}\tilde{S}_{n,+}(z))_+ \tag{2.20}$$

for all $j \in \mathbb{Z}_+$. Here $\tilde{S}_{n,-}(\lambda)$ and $\tilde{S}_{n,+}(z)$, $n \in \mathbb{Z}_N$, are the corresponding asymptotic expansions of the normalized monodromy matrix $\tilde{S}_n(\lambda) \in \mathscr{G}_-, n \in \mathbb{Z}_N$, as $\lambda \to \infty$ and as $\lambda = 1/z \to 0$, respectively.

In particular, making use of (2.10) we can normalize the monodromy matrix $\tilde{S}_n(\lambda)$, $n \in \mathbb{Z}_N$, by the asymptotic condition

$$S_n(\lambda) \overset{\lambda\to\infty}{\to} \tilde{S}_{n,-}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \lambda^{-1} + O(\lambda^{-2}) \in \mathcal{G}_-. \tag{2.21}$$

Correspondingly, as $z = 1/\lambda \to \infty$ one obtains infinite the asymptotic expansion

$$S_n(\lambda) \overset{\lambda=1/z\to 0}{\to} \tilde{S}_{n,+}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{-1} + O(z^{-2}) \in \mathcal{G}_-. \tag{2.22}$$

The hierarchies of evolution equations (2.19) and (2.20) can be rewritten as the following generating flows:

$$\frac{d}{dt_{(\mu_-)}} l_n[u,v;\lambda] = \frac{\lambda\mu_-}{\mu_- - \lambda}[\tilde{S}_{n+1,-}(\mu_-)l_n[u,v;\lambda] - l_n[u,v;\lambda]\tilde{S}_{n,-}(\mu_-)] \tag{2.23}$$

as $\lambda \to \infty$ and $|\lambda/\mu_-| < 1$,

$$\frac{d}{dt_{(\mu_+)}} l_n[u,v;1/z] = \frac{z\mu_+}{\mu_+ - z}[\tilde{S}_{n+1,+}(\mu_+)l_n[u,v;1/z] - l_n[u,v;1/z]\tilde{S}_{n,+}(\mu_+)] \tag{2.24}$$

as $z \to \infty$, and $|z_+/\mu_+| < 1$, where

$$\frac{d}{dt_{(\mu_-)}} = \sum_{j\in\mathbb{Z}_+} \mu_-^{-j}\frac{d}{dt_{j,-}}, \frac{d}{dt_{(\mu_+)}} = \sum_{j\in\mathbb{Z}_+} \mu_+^{-j}\frac{d}{dt_{j,+}}. \tag{2.25}$$

Let us now describe the analytical structure of the regularized matrices $\tilde{S}_{n,\pm}(\mu_\pm)$, $n \in \mathbb{Z}_N$, for arbitrary $\mu_\pm \in \mathbb{C}$. To do this we need to consider the exact expression for the monodromy matrices $S_{n,-}(\lambda)$, $\lambda \to \infty$, and $S_{n,+}(z)$, $z \to \infty$, defined by the expressions (1.17) and (1.18), respectively. Then we need to determine which of them together with the flows (2.23) and (2.24) makes it possible to find the corresponding evolution equations for the vectors $(\tilde{u}, \tilde{v})^\intercal \in M$, represented [17] by means of the related vertex operators [19] acting as follows:

$$\begin{aligned} &\hat{X}_\lambda^- : M \to M \text{ and } \hat{X}_z^+ : M \to M \\ &\hat{X}_\lambda^- := (\exp D_\lambda^-, \exp(-D_\lambda^-))^\intercal, \quad D_\lambda^- := \sum_{j\in\mathbb{Z}_+} \frac{1}{(j\lambda^j)}\frac{d}{dt_{j,-}} \end{aligned} \tag{2.26}$$

as $\lambda \to \infty$, and

$$\begin{aligned} &\hat{X}_\lambda^- : M \to M \text{ and } \hat{X}_z^+ : M \to M \\ &\hat{X}_\lambda^- := (\exp D_\lambda^-, \exp(-D_\lambda^-))^\intercal, \quad D_\lambda^- := \sum_{j\in\mathbb{Z}_+} \frac{1}{(j\lambda^j)}\frac{d}{dt_{j,-}}, \\ &\hat{X}_z^+ := (\exp D_z^+, \exp(-D_z^+))^\intercal, \quad D_z^- := \sum_{j\in\mathbb{Z}_+} \frac{1}{(jz^j)}\frac{d}{dt_{j,+}}, \end{aligned} \tag{2.27}$$

$$\begin{aligned} \hat{X}_\lambda^- \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} u(t_{0,-} + 1/\lambda, t_{0,+}; t_{1,-} + 1/(2\lambda^2), t_{1,+}; \ldots) \\ v(t_{0,-} - 1/\lambda, t_{0,+}; t_{1,-} - 1/(2\lambda^2), t_{1,+}; \ldots) \end{pmatrix}, \\ \hat{X}_z^+ \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} u(t_{0,-}, t_{0,+} + 1/z; t_{1,-}, t_{1,+} + 1/(2z^2); \ldots) \\ v(t_{0,-}, t_{0,+} - 1/z; t_{1,-}, t_{1,+} - 1/(2z^2); \ldots) \end{pmatrix} \end{aligned} \tag{2.28}$$

as $\lambda = 1/z \to 0$. The exact construction of the flows (2.28) will be analyzed in the next section by means of the gradient-holonomic approach devised in [14, 15, 24, 25].

3. Monodromy matrix analysis

The monodromy matrix representations (1.17) and (1.18) give rise to the expressions

$$S_{n,-}(\lambda) = \begin{pmatrix} \frac{\bar{e}_n^{(1)}(\lambda)+\lambda^{-2}\tilde{u}_n(\lambda)\tilde{v}_{n-1}(\lambda)e_n^{-(2)}(\lambda)}{1+\lambda^{-2}\tilde{u}_n(\lambda)\tilde{v}_{n-1}(\lambda)} & \frac{\lambda^{-1}\tilde{u}_n(\lambda)[\bar{e}_n^{(1)}(\lambda)-\bar{e}_n^{(2)}(\lambda)]}{1+\lambda^{-2}\tilde{u}_n(\lambda)\tilde{v}_{n-1}(\lambda)} \\ \frac{\lambda^{-1}\tilde{v}_{n-1}(\lambda)[\bar{e}_n^{(1)}(\lambda)-\bar{e}_n^{(2)}(\lambda)]}{1+\lambda^{-2}\tilde{u}_n(\lambda)\tilde{v}_{n-1}(\lambda)} & \frac{\bar{e}_n^{(1)}(\lambda)\lambda^{-2}\tilde{u}_n(\lambda)\tilde{v}_{n-1}(\lambda)+\bar{e}_n^{(2)}(\lambda)}{1+\lambda^{-2}\tilde{u}_n(\lambda)\tilde{v}_{n-1}(\lambda)} \end{pmatrix} \tag{3.1}$$

as $\lambda \to \infty$ and

$$S_{n,+}(z) = \begin{pmatrix} \frac{\bar{e}_n^{(1)}(z)+z^{-2}\tilde{u}_{n-1}(z)\tilde{v}_n(z)\bar{e}_n^{(2)}(z)}{1+z^{-2}\tilde{u}_{n-1}(z)\tilde{v}_n(z)} & \frac{-z^{-1}\tilde{u}_{n-1}(z)[\bar{e}_n^{(1)}(z)-\bar{e}_n^{(2)}(z)]}{1+z^{-2}\tilde{u}_{n-1}(z)\tilde{v}_n(z)} \\ \frac{-z^{-1}\tilde{v}_n(z)[\bar{e}_n^{(1)}(z)-\bar{e}_n^{(2)}(z)]}{1+z^{-2}\tilde{u}_{n-1}(z)\tilde{v}_n(z)} & \frac{\bar{e}_n^{(2)}(z)+z^{-2}\tilde{u}_{n-1}(z)\tilde{v}_n(z)\bar{e}_n^{(1)}(z)}{1+z^{-2}\tilde{u}_{n-1}(z)\tilde{v}_n(z)} \end{pmatrix} \tag{3.2}$$

as $\lambda = 1/z \to 0$, where

$$\begin{aligned} \bar{e}_n^{(1)}(\lambda) &:= \bar{e}_{n+N,n}^{(1)}(\lambda) = \prod_{k=n}^{n+N-1}(\lambda + \lambda^{-1}u_n\tilde{v}_{n-1}(\lambda)), \\ \bar{e}_n^{(2)}(\lambda) &:= \bar{e}_{n+N,n}^{(2)}(\lambda) = \prod_{k=n}^{n+N-1}(\lambda^{-1} - \lambda^{-1}v_n\tilde{u}_n(\lambda)) \end{aligned} \tag{3.3}$$

as $\lambda \to \infty$, $n \in \mathbb{Z}$, and

$$\begin{aligned} \bar{e}_n^{(1)}(z) &:= \bar{e}_{n+N,n}^{(1)}(z) = \prod_{k=n}^{n+N-1}(z^{-1} - z^{-1}u_n\tilde{v}_n(z)), \\ \bar{e}_n^{(2)}(z) &:= \bar{e}_{n+N,n}^{(2)}(z) = \prod_{k=n}^{n+N-1}(z + z^{-1}\tilde{u}_{n-1}(z)v_n), \end{aligned} \tag{3.4}$$

as $\lambda = 1/z \to 0$, $n \in \mathbb{Z}$.

Now let us consider the first monodromy matrix (3.1) and apply to it a regularization that will help eliminate the singular factors $\bar{e}_n^{(1)}(\lambda)$ and $\bar{e}_n^{(2)}(\lambda)$ as $\lambda \to \infty$. It is easy to observe that these factors satisfy the condition

$$\bar{e}_n^{(1)}(\lambda)\bar{e}_n^{(2)}(\lambda) = 1 \tag{3.5}$$

for all $n \in \mathbb{Z}$ and $\lambda \to \infty$. Moreover, each of these factors and the functionals $\det S_{n,-}(\lambda)$, $SpS_{n,-}(\lambda)$ are conservative:

$$\begin{aligned} \frac{d}{dt_-}\bar{e}_n^{(1)}(\lambda) &= 0 = \frac{d}{dt_-}\bar{e}_n^{(2)}(\lambda), \\ \frac{d}{dt_-}\det S_{n,-}(\lambda) &= 0 = \frac{d}{dt_-}SpS_{n,-}(\lambda) \end{aligned} \tag{3.6}$$

for all $\lambda \to \infty$. On the basis of these properties, one can construct the following regularized monodromy matrix:

$$\tilde{S}_{n,-}(\lambda) = [(S_{n,-}(\lambda) - \frac{\mathbf{1}}{2}SpS_{n,-}(\lambda)]k_n^{-1}(\lambda) + \frac{1}{2\lambda}\mathbf{1} \in \mathscr{G}_- \tag{3.7}$$

as $\lambda \to \infty$, where

$$k_n(\lambda) := (\bar{e}_n^{(1)}(\lambda) - \bar{e}_n^{(2)}(\lambda))\lambda \tag{3.8}$$

is an invariant regularizing factor. Thus, the expression (3.7) gives rise to the matrix

$$\tilde{S}_{n,-}(\lambda) = \begin{pmatrix} \frac{\lambda}{\lambda^2+\tilde{u}_n(\lambda)\tilde{v}_{n-1}(\lambda)} & \frac{\tilde{u}_n(\lambda)}{\lambda^2+\tilde{u}_n(\lambda)\tilde{v}_{n-1}(\lambda)} \\ \frac{\tilde{v}_{n-1}(\lambda)}{\lambda^2+\tilde{u}_n(\lambda)\tilde{v}_{n-1}(\lambda)} & \frac{\tilde{u}_n(\lambda)\tilde{v}_{n-1}(\lambda)}{(\lambda^2+\tilde{u}_n(\lambda)\tilde{v}_{n-1}(\lambda))\lambda} \end{pmatrix}, \tag{3.9}$$

holding as $\lambda \to \infty$. As a simple consequence from (3.9), we obtain the functional relationships

$$\frac{\lambda \tilde{S}_{n,-}^{(12)}(\lambda)}{\tilde{S}_{n,-}^{(11)}(\lambda)} = \tilde{u}_n(\lambda), \frac{\lambda \tilde{S}_{n,-}^{(21)}(\lambda)}{\tilde{S}_{n,-}^{(11)}(\lambda)} = \tilde{v}_{n-1}(\lambda), \tag{3.10}$$

allowing to find the evolutions $\frac{d}{dt_{\mu_-}}\tilde{u}_n(\lambda)$ and $\frac{d}{dt_{\mu_-}}\tilde{v}_n(\lambda)$, $n \in \mathbb{Z}_N$, using the monodromy evolution equation

$$\frac{d}{dt_{\mu_-}}\tilde{S}_{n,-}(\lambda) = \frac{\lambda\mu_-}{\mu_- - \lambda}[(\tilde{S}_{n,-}(\mu_-), \tilde{S}_{n,-}(\lambda)], \tag{3.11}$$

which readily follows from the evolution equation (2.23) as $\lambda \to \infty$. Namely, taking the limit $\mu_- \to \lambda$, $|\lambda/\mu_-| < 1$, of (3.11) we obtain that

$$\frac{d}{dt_-}\tilde{S}_{n,-}(\lambda) = \lambda^2[\frac{d}{d\lambda}\tilde{S}_{n,-}(\lambda), \tilde{S}_{n,-}(\lambda)], \tag{3.12}$$

as $\lambda \to \infty$, where

$$\frac{d}{dt_-} := \sum_{j\in\mathbb{Z}_+} \lambda^{-j}\frac{d}{dt_{j,-}}. \tag{3.13}$$

Now we can calculate the evolution of the vector $(\tilde{u}_n(\lambda), \tilde{v}_{n-1}(\lambda))^\intercal \in M$ for any $n \in \mathbb{Z}_N$:

$$\begin{aligned} \frac{d}{dt_-}\begin{pmatrix} \tilde{u}_n(\lambda) \\ \tilde{v}_{n-1}(\lambda) \end{pmatrix} &= \left(\frac{d}{dt_-}\left(\frac{\lambda \tilde{S}_{n,-}^{(12)}(\lambda)}{\tilde{S}_{n,-}^{(11)}(\lambda)}\right), \frac{d}{dt_-}\left(\frac{\lambda \tilde{S}_{n,-}^{(21)}(\lambda)}{\tilde{S}_{n,-}^{(11)}(\lambda)}\right)\right)^\intercal = \\ &= \left(-\lambda^2\frac{d}{d\lambda}\left(\frac{\lambda \tilde{S}_{n,-}^{(12)}(\lambda)}{\tilde{S}_{n,-}^{(11)}(\lambda)}\right), \lambda^2\frac{d}{d\lambda}\left(\frac{\lambda \tilde{S}_{n,-}^{(21)}(\lambda)}{\tilde{S}_{n,-}^{(11)}(\lambda)}\right)\right)^\intercal = \\ &= \left(-\lambda^2\frac{d}{d\lambda}\tilde{u}_n, \lambda^2\frac{d}{d\lambda}\tilde{v}_n\right)^\intercal, \end{aligned} \tag{3.14}$$

which easily reduces to the following vertex operator representation:

$$\begin{pmatrix} \tilde{u}(\lambda) \\ \tilde{v}(\lambda) \end{pmatrix} = \hat{X}_\lambda^- \begin{pmatrix} u(t_-) \\ v(t_-) \end{pmatrix} := \begin{pmatrix} u^+(t_-) \\ v^-(t_-) \end{pmatrix} = \begin{pmatrix} u(t_{0,-}+1/\lambda, t_{1,-}+1/(2\lambda^2), \dots) \\ v(t_{0,-}-1/\lambda, t_{1,-}-1/(2\lambda^2), \dots) \end{pmatrix} \tag{3.15}$$

for the generating temporal parameter $t_- \in \mathbb{C}^{\mathbb{Z}_+}$.

For the second monodromy matrix (3.2), we can apply the same procedure as above to obtain the regularization (3.7)

$$\tilde{S}_{n,+}(z) = [S_{n,+}(z) - \frac{\mathbf{1}}{2} Sp S_{n,+}(z)] k_n^{-1}(z) + \frac{1}{2z}\mathbf{1} \in \mathcal{G}_-, \tag{3.16}$$

where

$$k_n(z) := z(\bar{e}_n^{(1)}(z) - \bar{e}_n^{(2)}(z)) \tag{3.17}$$

is the corresponding invariant factor as $\lambda = 1/z \to 0$. As a result of (3.16,) we obtain the regularized monodromy matrix

$$\tilde{S}_{n,+}(z) = \begin{bmatrix} \frac{z}{z^2+\tilde{u}_{n-1}(z)\tilde{v}_n(\lambda)} & \frac{-\tilde{u}_{n-1}(z)}{z^2+\tilde{u}_{n-1}(z)\tilde{v}_n(z)} \\ \frac{-\tilde{v}_n(z)}{z^2+\tilde{u}_{n-1}(z)\tilde{v}_n(z)} & \frac{\tilde{u}_{n-1}(z)\tilde{v}_n(z)}{z^2+\tilde{u}_{n-1}(z)\tilde{v}_n(z)} \end{bmatrix} \tag{3.18}$$

which implies the functional relationships

$$\frac{-z\tilde{S}_{n,+}^{(12)}(z)}{\tilde{S}_{n,+}^{(11)}(z)} = \tilde{u}_{n-1}(z), \frac{-z\tilde{S}_{n,+}^{(21)}(z)}{\tilde{S}_{n,+}^{(11)}(z)} = \tilde{v}_n(z), \tag{3.19}$$

holding as $\lambda = 1/z \to 0$. Taking into account that owing to (2.24) the regularized matrix (3.18) satisfies the evolution equation

$$\frac{d}{dt_+}\tilde{S}_{n,+}(z) = [z^2\frac{d}{dz}\tilde{S}_{n,+}(z), \tilde{S}_{n,+}(z)], \tag{3.20}$$

we obtain

$$\begin{aligned} \frac{d}{dt_+}\begin{pmatrix} \tilde{u}_{n-1}(z) \\ \tilde{v}_n(z) \end{pmatrix} &= \left(-z\frac{d}{dt_+}\left(\frac{\tilde{S}_{n,+}^{(12)}(z)}{\tilde{S}_{n,+}^{(11)}(z)}\right), z\frac{d}{dt_+}\left(\frac{\tilde{S}_{n,+}^{(21)}(z)}{\tilde{S}_{n,+}^{(11)}(z)}\right)\right)^\intercal = \\ &= \left(-z^2\frac{d}{dz}\left(\frac{-z\tilde{S}_{n,+}^{(12)}(z)}{\tilde{S}_{n,+}^{(11)}(z)}\right), z^2\frac{d}{dz}\left(\frac{-z\tilde{S}_{n,+}^{(21)}(z)}{\tilde{S}_{n,+}^{(11)}(z)}\right)\right)^\intercal = \\ &= \left(-z^2\frac{d}{dz}\tilde{u}_{n-1}(z), z^2\frac{d}{d\lambda}\tilde{v}_n(z)\right)^\intercal, \end{aligned} \tag{3.21}$$

where

$$\frac{d}{dt_+} := \sum_{j\in\mathbb{Z}_+} z^{-j}\frac{d}{dt_{j,+}}, \tag{3.22}$$

is the generating vector field with respect to the temporal parameter $t_+ \in \mathbb{C}^{\mathbb{Z}_+}$. As a result of the differential relationships (3.21), we obtain the vertex operator representation for the vector $(\tilde{u}(z), \tilde{v}(z))^\intercal \in M$, namely

$$\begin{pmatrix} \tilde{u}(z) \\ \tilde{v}(z) \end{pmatrix} = \hat{X}_z^+ \begin{pmatrix} u(t_+) \\ v(t_+) \end{pmatrix} := \begin{pmatrix} u^+(t_+) \\ v^-(t_+) \end{pmatrix} = \begin{pmatrix} u(t_{0,+} + 1/z, t_{1,+} + 1/(2z^2), \ldots) \\ v(t_{0,+} - 1/z, t_{1,+} - 1/(2z^2), \ldots) \end{pmatrix}, \tag{3.23}$$

which holds as $\lambda = 1/z \to 0$.

Both vertex operator representations (3.15) and (3.23) can be combined as

$$\begin{pmatrix} \tilde{u}(\lambda;z) \\ \tilde{v}(\lambda;z) \end{pmatrix} = \hat{X}_\lambda^- \hat{X}_z^+ \begin{pmatrix} u(t_-;t_+) \\ v(t_-;t_+) \end{pmatrix} := \begin{pmatrix} u^+(t_-;t_+) \\ v^-(t_-;t_+) \end{pmatrix} = \tag{3.24}$$
$$= \begin{pmatrix} u(t_{0,-}+1/\lambda, t_{0,+}+1/z; t_{1,-}+1/(2\lambda^2), t_{1,+}+1/(2z^2);\ldots) \\ v(t_{0,-}-1/\lambda, t_{0,+}-1/z; t_{1,-}-1/(2\lambda^2), t_{1,+}-1/(2z^2);\ldots) \end{pmatrix},$$

and the following determining functional relationships

$$\begin{aligned} u_n^+(\lambda;z) &= u_n + \lambda^{-2} u_{n+1}^+(\lambda;z)(1 - v_n u_n^+(\lambda;z)), \\ v_n^+(\lambda;z) &= v_n + \lambda^{-2} v_{n-1}^-(\lambda;z)(1 - u_n v_n^-(\lambda;z)), \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} u_n^+(\lambda;z) &= u_n + z^{-2} u_{n-1}^+(\lambda;z)(1 - v_n u_n^+(\lambda;z)), \\ v_n^+(\lambda;z) &= v_n + z^{-2} v_{n+1}^-(\lambda;z)(1 - u_n v_n^-(\lambda;z)), \end{aligned} \tag{3.26}$$

hold, respectively, as $\lambda \to \infty$ and $z \to \infty$.

Now we shall analyze the matrix expression (1.10), making use of the vertex representation (3.24). Let us define the τ-function by means of the functional relationship

$$u_n := \rho_n/\tau_n, v_n := \sigma_n/\tau_n, \tag{3.27}$$

for some function $(\rho,\sigma)^\intercal : \mathbb{Z}_N \times \mathbb{C} \to \mathbb{C}^2$, holding for all $n \in \mathbb{Z}_N$, and represent the matrix (1.10) in the following slightly modified form:

$$\tilde{F}_{m,n}(z) = \begin{pmatrix} z^{n-m} e_{m,n}^{(1)}(z) & z^{m-1-n} u_{m-1}^+(z) e_{m,n}^{(2)}(z) \\ -z^{n-(m+1)} v_m^-(z) e_{m,n}^{(1)}(z) & z^{m-n} e_{m,n}^{(2)}(z) \end{pmatrix}, \tag{3.28}$$

where

$$e_{m,n}^{(1)}(z) := \prod_{k=n}^{m-1}(1 - u_k v_k^-(z)) = \frac{\prod_{k=0}^{m-1}(1 - u_k v_k^-(z))}{\prod_{k=0}^{n-1}(1 - u_k v_k^-(z))}, \tag{3.29}$$
$$e_{m,n}^{(2)}(z) := \prod_{k=n}^{m-1}(1 + z^{-2} v_k u_{k-1}^+(z)) = \frac{\prod_{k=0}^{m-1}(1 + z^{-2} v_k u_{k-1}^+(z))}{\prod_{k=0}^{n-1}(1 + z^{-2} v_k u_{k-1}^+(z))},$$

holds as $z \to \infty$. Using (3.29), one can define two functions $\alpha, \beta : \mathbb{Z}_N \times \mathbb{C} \to \mathbb{C}$, such that

$$1 - u_k v_k^-(z) = \alpha_{k+1}(z)/\alpha_k(z), \alpha_k(z) := \prod_{k=0}^{k-1}(1 - u_k v_k^-(z)), \tag{3.30}$$
$$e_{m,n}^{(1)}(z) = \prod_{k=n}^{m-1} \frac{\alpha_{k+1}(z)}{\alpha_k(z)} = \frac{\alpha_m(z)}{\alpha_n(z)},$$

$$1+z^{-2}v_k u_k^+(z) = \beta_k(z)/\beta_{k-1}(z), \tag{3.31}$$
$$\beta_k(z) = \prod_{k=0}^{n-1}(1+z^{-2}v_k u_k^+(z)),$$
$$e_{m,n}^{(2)}(z) = \prod_{k=n}^{m-1}\frac{\beta_k(z)}{\beta_{k-1}(z)} = \frac{\beta_{m-1}(z)}{\beta_{n-1}(z)}.$$

Now, taking into account the vertex operator relationships (3.26), one can derive the following functional equality from (3.30) and (3.31):

$$\beta_{k+1}^-(z)\alpha_k(z) = \beta_k(z)\alpha_k^+(z), \tag{3.32}$$

which holds for all $k \in \mathbb{Z}_N$. To resolve (3.32) regarding the mappings $\alpha, \beta : \mathbb{Z}_N \times \mathbb{C} \to \mathbb{C}$, we define the τ-function $\tau : \mathbb{Z}_N \times \mathbb{C} \to \mathbb{C}$ as

$$\tau_k(z) := \prod_{s=0}^{k-1}\beta_s(z)\alpha_s^+(z) \tag{3.33}$$

and observe that the expressions

$$\alpha_k(z) = \frac{\tau_{k+1}^-(z)}{\tau_k(z)}, \beta_k(z) = \frac{\tau_k^+(z)}{\tau_k(z)}, \tag{3.34}$$

solve the functional equation (3.32) for all $k \in \mathbb{Z}_N$. As a result of the exact expressions (3.34), (3.29) and (3.30), the matrix (3.28) is represented by means of the τ-function (3.33) as

$$\tilde{F}_{m,n}(z) = \begin{pmatrix} \frac{z^{n-m}\tau_m^-(z)\tau_n(z)}{\tau_{m-1}(z)\tau_{n-1}^-(z)} & \frac{z^{m-n-1}\rho_m^+(z)\tau_n(z)}{\tau_{m-1}(z)\tau_n^+(z)}) \\ \frac{-z^{n-m-1)}\sigma_m^-(z)\tau_{n-1}(z)}{\tau_{m-1}(z)\tau_{n-1}^-(z)} & \frac{z^{m-n}\tau_m^+(z)\tau_n(z)}{\tau_{m-1}(z)\tau_n^+(z)} \end{pmatrix} \tag{3.35}$$

which holds as $z \to \infty$ for all $n, m \in \mathbb{Z}_N$, where we have used (3.27).

The vertex operator representation of the matrix (3.35), solving the determining linear problem (1.5), is expressed by means of the τ-function (3.33) and coincides with the one obtained in [17,18] using a different approach. It is worth mentioning here that similar results can be obtained with respect to the second [20,21] linear spectral problem

$$f_{n+1} = l_n[u,v;\lambda]f_n, \quad l_n[u,v;\lambda] := \begin{pmatrix} \lambda & u_n \\ \lambda v_n & 1 \end{pmatrix}, \tag{3.36}$$

which possesses the only singularity as $\lambda \to \infty$ and generates functionally different vertex operator representations for solutions to the related Ablowitz-Ladik integrable hierarchy and for the corresponding τ-representation of the matrix solution of the linear problem (1.3) and (1.4). These and related matters shall be considered in detail in forthcoming papers.

4. Hamiltonian analysis

The Poisson structure (2.18) on the phase space M_G can be naturally reduced on the discrete manifold M, giving rise to the Poisson bracket

$$\{\tilde{\gamma},\tilde{\xi}\}_{\vartheta} := (<\operatorname{grad}\tilde{\gamma}[u,v], \vartheta \operatorname{grad}\tilde{\xi}[u,v]>) \tag{4.1}$$

for any smooth functionals $\tilde{\gamma},\tilde{\xi} \in \mathscr{D}(M)$, where the standard convolution on $T^*(M)\times T(M)$ is denoted by $(<\cdot,\cdot>)$ and

$$\vartheta_n := \begin{pmatrix} 0 & h \\ -h_n & 0 \end{pmatrix}, \quad h_n := 1-u_n v_n, \tag{4.2}$$

for $n\in\mathbb{Z}$, $\vartheta : T^*(M)\to T(M)$ is an implectic operator. Now, making use of the gradient-holonomic scheme devised in [14, 24–26, 28], one finds that

$$\operatorname{grad}(SpS_N(\lambda))_n[u,v] = \begin{pmatrix} \lambda^{-1}h_n^{-1}S_n^{(21)}(\lambda) - h_n^{-1}v_n S_n^{(22)}(\lambda) \\ \lambda h_n^{-1}S_n^{(12)}(\lambda) - h_n^{-1}u_n S_n^{(11)}(\lambda) \end{pmatrix} \tag{4.3}$$

is satisfied for all $n\in\mathbb{Z}$ and $\lambda\in\mathbb{C}$. Moreover, one deduces the important gradient relationship

$$\lambda\vartheta\operatorname{grad}(SpS_N(\lambda))[u,v] = \eta\operatorname{grad}(SpS_N(\lambda))[u,v], \tag{4.4}$$

where

$$\eta_n := \begin{pmatrix} (h_n - u_n D_n^{-1}u_n)\Delta & (u_n^2 + u_n D_n^{-1}u_n)\Delta^{-1} \\ v_n D_n^{-1} v_n \Delta & -(1+v_n D_n^{-1}u_n)\Delta^{-1} \end{pmatrix} \begin{pmatrix} u_n D_n^{-1}u_n & h_n - u_n D_n^{-1}u_n \\ v_n D_n^{-1}v_n & -(v_n + v_n D_n^{-1}v_n) \end{pmatrix} \tag{4.5}$$

and for any $n\in\mathbb{Z}$ the operations are defined as

$$\begin{aligned} D_n^{-1}(\ldots) :=& \frac{1}{2}\left(\sum_{k=n}^{n+m-1}(\ldots)_k - \sum_{k=n+m}^{N+n-1}(\ldots)\right), \\ \Delta(\ldots)_n :=& (\ldots)_{n+1}, D_n(\ldots) := (\Delta-1)(\ldots)_n. \end{aligned} \tag{4.6}$$

As a simple consequence of the gradient relationship (4.4), we obtain that all of the Casimir invariants $\gamma,\xi\in I(\mathscr{G}_N^*)$ constructed above, when reduced on the discrete manifold M, are in involution with respect to both the first Poisson structure (4.1) and the second Poisson structure

$$\{\tilde{\gamma},\tilde{\xi}\}_{\vartheta} := (<\operatorname{grad}\tilde{\gamma}[u,v], \eta \operatorname{grad}\tilde{\xi}[u,v]>) \tag{4.7}$$

for any smooth functionals $\tilde{\gamma},\tilde{\xi}\in\mathscr{D}(M)$ on the manifiold M. In particular, the relationship (4.4) implies the Magri compatibility [22] of the implectic structures (4.2) and (4.5) on the whole manifold M.

Now, making use of the gradient expression (4.3), we can show that the generating vector fields 3.13) and (3.22) have Hamiltonian representations of the form

$$\frac{d}{dt_-}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = -\vartheta_n \lambda^2 \mathrm{grad} SpS_{N,-}(\lambda)_n[u,v] = \begin{pmatrix} -\lambda^2 S_{n,-}^{(12)}(\lambda) + \lambda u_n S_{n,-}^{(11)}(\lambda) \\ \lambda S_n^{(21)}(\lambda) - \lambda^2 v_n S_n^{(22)}(\lambda) \end{pmatrix}, \tag{4.8}$$

holding as $\lambda \to \infty$, and

$$\frac{d}{dt_+}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = -\vartheta_n z \mathrm{grad} SpS_{N,+}(z)_n[u,v] = \begin{pmatrix} -S_{n,+}^{(12)}(z) + z u_n S_{n,+}^{(11)}(z) \\ z^2 h_n^{-1} S_n^{(21)}(\lambda) - z v_n S_{n,+}^{(22)}(z) \end{pmatrix}, \tag{4.9}$$

holding as $\lambda = 1/z \to 0$. It is easy to check that the generating evolution equations (4.8) and (4.9) are completely compatible with the vertex operator representation (3.24) and the related generating functional relationships (3.25) and (3.26). To be more precise, one can find from (3.25) and (3.26) that

$$\begin{aligned}
&\frac{d}{dt_{0,+}}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} h_n u_n \\ -h_n v_n \end{pmatrix}, \ \frac{d}{dt_{0,-}}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} -u_n v_{n+1} \\ -u_n u_{n-1} \end{pmatrix}, \\
&\frac{d}{dt_{2,-}}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} -d^2 u_n/dt_{0,-}^2 + 2u_n u_{n-1} dv_n/dt_{0,-} \\ d^2 v_n/dt_{0,-}^2 + 2v_n v_{n+1} du_n/dt_{0,-} \end{pmatrix}, \ \frac{d}{dt_{2j+1,\pm}}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = 0, \\
&\frac{d}{dt_{4,-}}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u_{n-2}h_{n-2}h_n - u_{n-1}^2 h_n v_n - u_{n-1} h_n u_n v_{n+1} \\ v_{n-1} h_n h v_n v_{n+1} + v_{n+1}^2 h_n u_n - h_n h_{n+1} v_{n+2} \end{pmatrix},
\end{aligned} \tag{4.10}$$

for all $j \in \mathbb{Z}_+$ and so on.

5. Conclusion

The approach devised herein for studying the infinite hierarchy of the Ablowitz–Ladik integrable differential-difference dynamical systems and their vertex operator representation appears to be sufficiently regular and can be effective for constructing vertex representations of integrable dynamical systems on discrete manifolds. It is also worth mentioning the close relationship between the approach presented and the τ-function representation of the fundamental solution to the corresponding linear discrete Lax type problem, representing the solutions to the Ablowitz–Ladik integrable differential-difference dynamical systems. In particular, the analytical scheme of this work clarifies some vague and lengthy calculations in [17–19], and can be successfully applied to other hierarchies of Lax integrable nonlinear dynamical systems on discrete manifolds.

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