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Research Article

Inverse Scattering Transformation for the Fokas–Lenells Equation with Nonzero Boundary Conditions

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ABSTRACT

In this article, we focus on the inverse scattering transformation for the Fokas–Lenells (FL) equation with nonzero boundary conditions via the Riemann–Hilbert (RH) approach. Based on the Lax pair of the FL equation, the analyticity, symmetry and asymptotic behavior of the Jost solutions and scattering matrix are discussed in detail. With these results, we further present a generalized RH problem, from which a reconstruction formula between the solution of the FL equation and the Riemann–Hilbert problem is obtained. The *N*-soliton solutions of the FL equation is obtained by solving the RH problem.

1. INTRODUCTION

The Nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} - 2v|q|^2 q = 0 \tag{1.1}$$

with $v = \pm 1$ is a significant mathematical and physical model for the optical fibers, deep water waves and plasma physics [3]. The NLS equation is a completely integrable system which admits Lax pair and bi-Hamiltonian formulation [21,22].

In the late 70s, standard bi-Hamiltonian formulation was used to obtain an integrable generalization of a given equation [7]. For instance, the Camassa–Holm equation is derived from the KdV equation via the bi-Hamiltonian structure [6], and the same mathematical trick can be applied to the NLS equation yields Fokas–Lenells (FL) equation [8]

$$iu_t - vu_{tx} + \gamma u_{xx} + \sigma |u|^2 (u + ivu_x) = 0, \sigma = \pm 1. \tag{1.2}$$

If let $\alpha = \gamma/v > 0$ and $\beta = \dfrac{1}{v}$ and

$$u \to \beta\sqrt{\alpha} e^{i\beta x} u, \sigma \to -\sigma,$$

then equation (1.2) can be converted into

$$u_{tx} + \alpha\beta^2 u - 2i\alpha\beta u_x - \alpha u_{xx} + \sigma i\alpha\beta^2 |u|^2 u_x = 0, \tag{1.3}$$

where $\alpha, \beta > 0$. In this paper, we consider the focusing case with $\sigma = -1$.

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In recent years, much work has been done on the FL equation (1.3). For example, Lax pair for FL equation was obtained via the bi-Hamiltonian structure by Fokas and Lenells [8]. They further considered the initial-boundary problem for the FL equation (1.3) on the half-line by using the Fokas unified method [9]. The dressing method is applied to obtain an explicit formula for bright-soliton and dark soliton solutions for Eq. (1.3) [11,15]. The bilinear method was used to obtain bright and dark soliton solution are obtained [12,13]. The Darboux transformation is used to obtain rogue waves of the FL equation [10,19]. The Deift–Zhou nonlinear steepest decedent method was used to analyze long-time asymptotic behavior for FL equation with decaying initial value [18]. Riemann–Hilbert (RH) approach was adopted to construct explicit soliton solutions under zero boundary conditions [1]. As far as we know, the soliton solutions for the FL equation (1.3) with Nonzero Boundary Conditions (NZBCs) have not been reported. In this paper, we apply RH approach to study the inverse transformation of the FL equation (1.3) with the following NZBCs

$$u(x,t)\sim q_{\pm}e^{-i\beta x+2i\alpha\beta t},\ x\to\pm\infty, \tag{1.4}$$

where $|q_{\pm}|=\sqrt{\dfrac{2}{\beta}}$ and assume that $u(x,t)-q_{\pm}\in L^1(R\pm)$; with respect to x for all $t\geq 0$. In next section, we will see that this kind of NZBCs (1.4) avoids the discussion on a multi-valued function as the case of the NLS equation [3].

The inverse scattering transform is an important method to study important nonlinear wave equations with Lax pair such as the NLS equation, the modified KdV equation, Sine-Gordan equation [5,14]. As an improved version of inverse scattering transform, the RH method has been widely adopted to solve nonlinear integrable systems [2,4,16,17,20,23].

The paper is organized as follows. In Section 2, by introducing appropriate transformations, we change the asymptotic boundary conditions (1.4) into constant boundary conditions. Furthermore, we analyze the analyticity, symmetry and asymptotic behavior of eigenfunctions and scattering matrix associated with the Lax pair. In Section 3, a generalized RH problem for the FL equation is constructed, and the distribution of discrete spectrum and residue conditions associated with RH problem are discussed. Based on these results, we reconstruct the potential function from the solution of the RH problem. In Section 4, we give the N-soliton solutions via solving RH problem under reflectionless case.

2. THE DIRECT SCATTERING WITH NZBCs

2.1. Jost Solutions

It is well-known that the FL equation (1.3) admits a Lax pair

$$\begin{aligned}&\psi_x+ik^2\sigma_3\psi=kU_x\psi,\\&\psi_t+i\eta^2\sigma_3\psi=\left[\alpha kU_x+\frac{i\alpha\beta^2}{2}\sigma_3\left(\frac{1}{k}U-U^2\right)\right]\psi,\end{aligned} \tag{2.1}$$

where

$$U=\begin{pmatrix}0&u\\v&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \eta=\sqrt{\alpha}\left(k-\frac{\beta}{2k}\right),\quad v=-u^*.$$

In order to invert the nonzero boundary conditions into the constant boundary conditions, we introduce a transformation for eigenfunctions and potentials.

Theorem 2.1. By transformation

$$\begin{aligned}&u=qe^{-i\beta x+2i\alpha\beta t},\\&\psi=e^{\left(-\frac{1}{2}i\beta x+i\alpha\beta t\right)\sigma_3}\varphi,\end{aligned} \tag{2.2}$$

the FL equation (1.3) becomes

$$q_{tx}-i\beta q_t+2i\alpha\beta q_x-\alpha q_{xx}+(2\alpha\beta^2-\alpha\beta^3|q|^2)q-i\alpha\beta^2|q|^2q_x=0 \tag{2.3}$$

with corresponding boundary conditions

$$q\to q_{\pm},\quad x\to\pm\infty. \tag{2.4}$$

And Eq. (2.3) is the compatibility condition of the Lax pair

$$\varphi_x=X\varphi,\quad \varphi_t=T\varphi, \tag{2.5}$$

where

$$X=-ik^2\sigma_3+\frac{1}{2}i\beta\sigma_3-i\beta k\sigma_3 Q+kQ_x,$$
$$T=-i\eta^2\sigma_3-\frac{1}{2}i\alpha\beta^2\sigma_3 Q^2-\frac{1}{2}i\alpha\beta^2 q_0^2\sigma_3-i\alpha\beta k\sigma_3 Q+\alpha kQ_x+\frac{i\alpha\beta^2}{2k}\sigma_3 Q,$$

with $Q=\begin{pmatrix}0 & q\\ -q^* & 0\end{pmatrix}$.

Proof. Suppose the transformation is

$$u=qe^{Aix+Bit},\ \psi=e^{(iCx+iDt)\sigma_3}\varphi.$$

Substituting it into (1.3) yields

$$q_{xt}+Aiq_t+i(B-2\alpha\beta-2\alpha A)q_x+(2\alpha\beta A+\alpha\beta^2+\alpha A^2-AB)q-\alpha q_{xx}+A\alpha\beta^2|q|^2q-i\alpha\beta^2|q|^2q_x=0. \tag{2.6}$$

And the Lax pair becomes

$$\begin{aligned}\varphi_x&=e^{\left[i\left(\frac{A}{2}-C\right)x+i\left(\frac{B}{2}-D\right)t\right]\hat{\sigma}_3}\begin{pmatrix}-ik^2-iC & k(Aiq+q_x)\\ k(Aiq^*-q_x^*) & ik^2+iC\end{pmatrix}\varphi,\\ \varphi_t&=e^{\left[i\left(\frac{A}{2}-C\right)x+i\left(\frac{B}{2}-D\right)t\right]\hat{\sigma}_3}\begin{pmatrix}-i\eta^2+\frac{1}{2}i\alpha\beta^2|q|^2-iD & \alpha k(Aiq+q_x)+\frac{i\alpha\beta^2}{2k}q\\ -\alpha k(-Aiq^*+q_x^*)+\frac{i\alpha\beta^2}{2k}q^* & i\eta^2-\frac{1}{2}i\alpha\beta^2|q|^2+iD\end{pmatrix}\varphi.\end{aligned} \tag{2.7}$$

Take $A=2C$ and $B=2D$ in the above equation and consider the limit as $x\to\pm\infty$, one gets

$$X_\pm=\begin{pmatrix}-ik^2-\frac{A}{2}i & kAiq_\pm\\ kAiq_\pm^* & ik^2+\frac{A}{2}i\end{pmatrix},$$
$$T_\pm=\begin{pmatrix}-i\eta^2+i\alpha\beta-iD & \alpha kiAq_\pm+\frac{i\alpha\beta^2}{2k}q_\pm\\ \alpha kiAq_\pm^*+\frac{i\alpha\beta^2}{2k}q_\pm^* & i\eta^2-i\alpha\beta+iD\end{pmatrix}.$$

These two matrices are proportional if

$$A=-\beta,\ B=2\alpha\beta.$$

Moreover, $T_\pm$ and $X_\pm$ are proportional

$$T_\pm=\frac{\sqrt{\alpha}\eta}{k}X_\pm.$$

which implies that their eigenvalues are proportional and they share the same eigenvector matrices.

To diagonalize the matrices $X_\pm$ and $T_\pm$ and further obtain the Jost solutions, we need to get the eigenvalues and eigenvector matrices of them. Direct calculation shows that the eigenvalues of the matrices $X_\pm$ are $\pm\frac{ik}{\sqrt{\alpha}}\lambda$; and the eigenvalues of the matrices $T_\pm$ are $\pm i\eta\lambda$, where

$$\lambda=\sqrt{\alpha}\left(k+\frac{\beta}{2k}\right),$$

and the corresponding eigenvector matrices are

$$Y_\pm(k)=\mathrm{I}-\frac{\beta}{2k}Q_\pm,\ \det Y_\pm(k)=1+\frac{\beta}{2k^2}\triangleq\gamma(k),$$

with $Q_{\pm} = \begin{pmatrix} 0 & q_{\pm} \\ -q_{\pm}^{*} & 0 \end{pmatrix}$.

With the above results, the matrices can be diagonalized as:

$$\begin{aligned} X_{\pm} &= Y_{\pm}\left(-\frac{ik\lambda\sigma_3}{\sqrt{\alpha}}\right)Y_{\pm}^{-1}, \\ T_{\pm} &= Y_{\pm}(-i\eta\lambda\sigma_3)Y_{\pm}^{-1}. \end{aligned} \tag{2.8}$$

Thus, the asymptotic Lax pair

$$\widetilde{\varphi}_x = X_{\pm}\widetilde{\varphi}, \qquad \widetilde{\varphi}_t = T_{\pm}\widetilde{\varphi} \tag{2.9}$$

can be written as

$$(Y_{\pm}^{-1}\widetilde{\varphi})_x = -\frac{ik\lambda\sigma_3}{\sqrt{\alpha}}(Y_{\pm}^{-1}\widetilde{\varphi}), \ (Y_{\pm}^{-1}\widetilde{\varphi})_t = -i\eta\sigma_3(Y_{\pm}^{-1}\widetilde{\varphi}),$$

which has a solution

$$\widetilde{\varphi} = Y_{\pm}(k)e^{-i\theta(x,t,k)\sigma_3}. \tag{2.10}$$

Therefore, the asymptotic of the eigenfunctions $\varphi_{\pm}$ are

$$\varphi_{\pm} \sim Y_{\pm}(k)e^{-i\theta(x,t,k)\sigma_3}, \ x \to \pm\infty, \tag{2.11}$$

where

$$\theta(x,t,k) = \frac{k\lambda x}{\sqrt{\alpha}} + \lambda\eta t.$$

Define the Jost solutions as

$$J_{\pm}(x,t,k) = \varphi_{\pm}(x,t,k)e^{i\theta\sigma_3}, \tag{2.12}$$

then the Lax pair (2.5) is changed to

$$\begin{aligned} J_{\pm,x} - \frac{ik\lambda}{\sqrt{\alpha}}J_{\pm}\sigma_3 &= XJ_{\pm}, \\ J_{\pm,t} - i\lambda\eta J_{\pm}\sigma_3 &= TJ_{\pm}, \end{aligned} \tag{2.13}$$

and

$$J_{\pm} \sim Y_{\pm}(k), \qquad x \to \pm\infty. \tag{2.14}$$

2.2. Scattering Matrix

The functions $\varphi_{\pm}$ are both fundamental matrix solutions of the Lax pair (2.5), thus there exists a matrix $S(k)$ that only depends on k, such that

$$\varphi_{+}(x,t,k) = \varphi_{-}(x,t,k)\,S(k), \tag{2.15}$$

where $S(k)$ is called scattering matrix. Columnwise, Eq. (2.15) reads

$$\varphi_{+,1} = s_{11}\varphi_{-,1} + s_{21}\varphi_{-,2}, \ \varphi_{+,2} = s_{12}\varphi_{-,1} + s_{22}\varphi_{-,2}. \tag{2.16}$$

Since $\mathrm{tr}(X) = \mathrm{tr}(T) = 0$, according to the Abel's formula, we have

$$(\det \varphi_{\pm})_x = (\det \varphi_{\pm})_t = 0,$$

which implies

$$\det \varphi_{\pm} = \lim_{x\to\pm\infty} \det \varphi_{\pm} = \gamma(k),$$

then the scattering coefficients can be expressed as Wronskians of columns $\varphi_\pm$ in the following way

$$s_{11}(k)=\frac{Wr(\varphi_{+,1},\varphi_{-,2})}{\gamma(k)},\qquad s_{12}(k)=\frac{Wr(\varphi_{+,2},\varphi_{-,2})}{\gamma(k)},$$
$$s_{21}(k)=\frac{Wr(\varphi_{-,1},\varphi_{+,1})}{\gamma(k)},\qquad s_{22}(k)=\frac{Wr(\varphi_{-,1},\varphi_{+,2})}{\gamma(k)}. \tag{2.17}$$

2.3. Asymptotic Analysis

To properly construct the Riemann–Hilbert problem, we need to consider the asymptotic behavior of eigenfunctions and scattering matrix as $k\to\infty$ and $k\to 0$.

2.3.1. Asymptotic as $k\to\infty$

Consider a solution of (2.13) of the form

$$J_\pm = J_\pm^{(0)}+\frac{J_\pm^{(1)}}{k}+\frac{J_\pm^{(2)}}{k^2}+\cdots,$$

then substituting the above expansion into (2.13) and comparing the coefficients of the same order of k, we get the comparison results:

x-part:

$$O(k^0): J_{\pm,x}^{(0)}+i[\sigma_3,J_\pm^{(2)}]=i\beta\sigma_3J_\pm^{(0)}-i\beta\sigma_3QJ_\pm^{(1)}+Q_xJ_\pm^{(1)}, \tag{2.18}$$
$$O(k^2): J_\pm^{(0)}\sigma_3=\sigma_3J_\pm^{(0)}. \tag{2.19}$$

t-part:

$$O(k):[\sigma_3,J_\pm^{(1)}]=-\beta\sigma_3QJ_\pm^{(0)}-iQ_xJ_\pm^{(0)}, \tag{2.20}$$
$$O(k^0): J_{\pm,t}^{(0)}+i\alpha[\sigma_3,J_\pm^{(2)}]=i\alpha\beta\sigma_3J_\pm^{(0)}-\frac{1}{2}i\alpha\beta^2Q^2\sigma_3J_\pm^{(0)}-\frac{1}{2}i\alpha\beta^2q_0^2\alpha_3J_\pm^{(0)}-i\alpha\beta\sigma_3QJ_\pm^{(1)}+\alpha Q_xJ_\pm^{(1)}. \tag{2.21}$$

Based on these results, we derive that $J_\pm^{(0)}$ is diagonal and satisfies

$$J_{\pm,x}^{(0)}=iv_1\sigma_3J_\pm^{(0)}, \tag{2.22}$$
$$J_{\pm,t}^{(0)}=iv_2\sigma_3J_\pm^{(0)}, \tag{2.23}$$

where

$$v_1(x,t)=\beta+\frac{1}{2}\beta^2qr-\frac{1}{2}i\beta qr_x+\frac{1}{2}i\beta q_xr+\frac{1}{2}q_xr_x,$$
$$v_2(x,t)=\alpha\beta-\frac{1}{2}\alpha\beta^2q_0^2+\frac{1}{2}i\alpha\beta(q_xr-qr_x)+\frac{1}{2}\alpha q_xr_x.$$

Note that the FL equation (2.3) admits the conservation law

$$\left(q_xr_x-i\beta qr_x+i\beta q_xr+\beta^2qr\right)_t=\left(\alpha q_xr_x+i\alpha\beta(q_xr-qr_x)\right)_x,$$

thus Eqs. (2.22) and (2.23) are consistent and are both satisfied if we define

$$J_\pm^{(0)}=e^{iv\sigma_3},\ v=\int_{-\infty}^{x}\left(\beta+\frac{1}{2}\beta^2qr-\frac{1}{2}i\beta qr_x+\frac{1}{2}i\beta q_xr+\frac{1}{2}q_xr_x\right)dx'. \tag{2.24}$$

Therefore, we obtain the limit of the Jost solutions as $k\to\infty$:

$$J_\pm\sim J_\pm^{(0)},\ k\to\infty. \tag{2.25}$$

Define

$$J_\pm=J_\pm^{(0)}\mu_\pm, \tag{2.26}$$

then we have

$$\mu_{\pm} \to \mathrm{I}, \quad k \to \infty. \tag{2.27}$$

In addition, the asymptotic property of $\varphi_{\pm}$ follows:

$$\varphi_{\pm} \sim e^{i(\nu-\theta)\sigma_3}, \quad k \to \infty.$$

Consider the wronskians expression of the scattering coefficients (2.17), we find that

$$s_{11}(k) \to 1,\, s_{22}(k) \to 1, \quad k \to \infty. \tag{2.28}$$

2.3.2. Asymptotic as k → 0

We first assume that $J_{\pm}$ admit a Lorrent expansion

$$J_{\pm} = \sum_{n=-\infty}^{n=\infty} k^n D_{\pm}^{(n)}.$$

Substituting into (2.13) and comparing the coefficients of the same order of k gives that $D_{\pm}^{(n)} = 0, n = -2, -3, \cdots$. Thus we expend solution of (2.13) of the form

$$J_{\pm} = \frac{D_{\pm}^{(-1)}}{k} + D_{\pm}^{(0)} + D_{\pm}^{(1)}k + \cdots,$$

then we obtain the comparison results:

x-part:

$$O(k^{-1}): D_{\pm,x}^{(-1)} - \frac{1}{2}i\beta D_{\pm}^{(-1)}\sigma_3 = \frac{1}{2}i\beta\sigma_3 D_{\pm}^{-1}, \tag{2.29}$$

t-part:

$$O(k^{-3}): D_{\pm}^{-1}\sigma_3 = -\sigma_3 D_{\pm}^{-1}. \tag{2.30}$$

It is easy to check that

$$D_{\pm,x}^{(-1)} = 0, \tag{2.31}$$

which implies $D_{\pm}^{(-1)}$ is a constant independent on x. This implies that the following limit

$$\lim_{k\to 0} kJ_{\pm} = D_{\pm}^{(-1)} \tag{2.32}$$

is uniformly convergent for $x \in R$. While the following limit

$$\lim_{x\to\pm\infty} kJ_{\pm} = kI - \frac{\beta}{2}\mathrm{Q}_{\pm} \tag{2.33}$$

exists for every fixed $k \in C$. Thus the following limit is commutative, and using (2.32) and (2.33) yields

$$D_{\pm}^{(-1)} = \lim_{x\to\pm\infty}\lim_{k\to 0} kJ_{\pm} = \lim_{k\to 0}\lim_{x\to\pm\infty} kJ_{\pm} = -\frac{\beta}{2}\mathrm{Q}_{\pm},$$

which leads to

$$J_{\pm} = -\frac{\beta \mathrm{Q}_{\pm}}{2k} + O(1), \quad k \to 0. \tag{2.34}$$

The asymptotics of functions $\varphi_{\pm}$ and $\mu_{\pm}$ can be obtained by transformations (2.12) and (2.26). Consider the wronskian expressions of scattering coefficients and note the boundary conditions, we find that

$$s_{11} = Wr(J_{+,1}e^{-i\theta}, J_{-,2}e^{i\theta}) = \frac{q_-}{q_+} + O(1), \;\; k \to 0,$$
$$s_{22} = Wr(J_{-,1}e^{-i\theta}, J_{+,2}e^{i\theta}) = \frac{q_-^*}{q_+^*} + O(1), \;\; k \to 0.$$

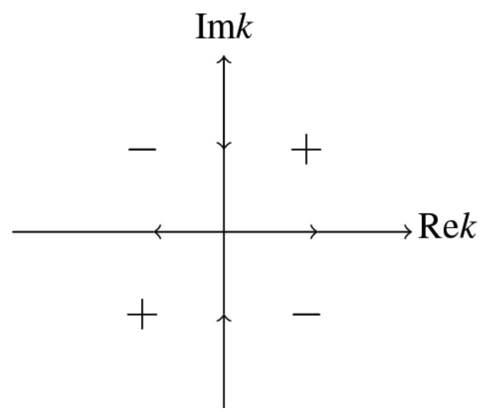

Figure 1 | The jump contour in the complex k-plane.

2.4. Analyticity

Noticing that

$$\mathrm{Im}\left(\frac{k\lambda}{\sqrt{\alpha}}\right) = 4\mathrm{Re}k\mathrm{Im}k,$$

we define two domains and their boundary

$$\begin{aligned}
D^{+} &= \{k \,|\, \mathrm{Re}k\mathrm{Im}k > 0\} = \left\{k \,|\, \arg k \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3}{2}\pi\right)\right\},\\
D^{-} &= \{k \,|\, \mathrm{Re}k\mathrm{Im}k < 0\} = \left\{k \,|\, \arg k \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3}{2}\pi, 2\pi\right)\right\},\\
\Sigma &= \{k \,|\, \mathrm{Re}k\mathrm{Im}k = 0\} = R \cup iR.
\end{aligned}$$

which are shown in Figure 1.

Under the transformation (2.26), we write the Lax pair as

$$\begin{aligned}
&\mu_{\pm,x} - \frac{ik\lambda}{\sqrt{\alpha}}[\mu_{\pm}, \sigma_3] = \widehat{X}\mu_{\pm},\\
&\mu_{\pm,t} - i\lambda\eta[\mu_{\pm}, \sigma_3] = \widehat{T}\mu_{\pm},
\end{aligned} \tag{2.35}$$

where

$$\begin{aligned}
\widehat{X} &= e^{-iv\hat{\sigma}_3}\left(X + \frac{ik\lambda\sigma_3}{\sqrt{\alpha}} - iv_1\sigma_3\right),\\
\widehat{T} &= e^{-iv\hat{\sigma}_3}(T + i\lambda\eta\sigma_3 - iv_2\sigma_3).
\end{aligned}$$

We rewrite the Eq. (2.35) into a full differential form

$$d(e^{i\theta\hat{\sigma}_3}\mu_{\pm}) = e^{i\theta\hat{\sigma}_3}(\widehat{X}dx + \widehat{T}dt)\,\mu_{\pm}, \tag{2.36}$$

which implies $\mu_{\pm}$ can be formally integrated to obtain the integral equations for the eigenfunctions:

$$\mu_{-} = \mu_{-}^{0} + \int_{-\infty}^{x} e^{-\frac{ik\lambda}{\sqrt{\alpha}}(x-y)\hat{\sigma}_3}\widehat{X}(y,t,k)\mu_{-}(y,t,k)\,dy, \tag{2.37}$$

$$\mu_{+} = \mu_{+}^{0} - \int_{x}^{+\infty} e^{-\frac{ik\lambda}{\sqrt{\alpha}}(x-y)\hat{\sigma}_3}\widehat{X}(y,t,k)\mu_{+}(y,t,k)\,dy, \tag{2.38}$$

where

$$\mu_{-}^{0} = Y_{-}, \quad \mu_{+}^{0} = e^{-i\int_{-\infty}^{+\infty} v_1(x')dx'\sigma_3}Y_{+}.$$

By using a similar way to Appendix A in Biondini and Kovačič [1], under mild integrability conditions on the potential, the eigenfunctions (2.37) and (2.38) can be analytically extended in the complex k-plane into the following regions:

$$\mu_{-,1},\ \mu_{+,2}: D^{+},\ \mu_{+,1},\ \mu_{-,2}: D^{-}, \tag{2.39}$$

where $\mu_{\pm} = (\mu_{\pm,1}, \mu_{\pm,2})$, the subscript 1, 2 denote the first and second column of $\mu_{\pm}$.

Apparently, the functions $\varphi_{\pm}$ and $\mu_{\pm}$ share the same analyticity, hence from the wronskians expression of the scattering coefficients, we know that s_{11} is analytic in D^{-}, and s_{22} is analytic in D^{+}.

2.5. Symmetry

To investigate the discrete spectrum and residue conditions in the Riemann–Hilbert problem, one needs to analyze the symmetric property for the solutions $\varphi_{\pm}$ and the scattering matrix $S(k)$.

Theorem 2.2. The Jost eigenfunctions satisfy the following symmetric relations

$$\sigma\varphi_{\pm}^{*}(k^{*})\sigma = -\varphi_{\pm}(k), \tag{2.40}$$

$$\sigma_1\varphi_{\pm}^{*}(-k^{*})\sigma_1 = \varphi_{\pm}(k), \tag{2.41}$$

and the scattering matrix satisfies

$$-\sigma S(k)\sigma = S^{*}(k^{*}), \tag{2.42}$$

$$\sigma_1 S^{*}(-k^{*})\sigma_1 = S(k). \tag{2.43}$$

where

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. We only prove (2.40) and (2.42), (2.41) and (2.43) can be shown in a similar way. The functions $\varphi_{\pm}$ are the solutions of the spectral problem

$$\varphi_{\pm,x}(k) = X(k)\varphi_{\pm}(k). \tag{2.44}$$

Conjugating and multiplying left by σ on both sides of the equation gives

$$\left(\sigma\varphi_{\pm}^{*}(k^{*})\sigma\right)_x = \sigma X^{*}(k^{*})\varphi_{\pm}^{*}(k^{*})\sigma.$$

Note that $\sigma X^{*}(k^{*}) = X(k)\sigma$, hence $\sigma\varphi_{\pm}^{*}(k^{*})\sigma$ is also a solution of (2.44). By using relations $\sigma Y_{\pm}^{*}(k^{*}) = Y_{\pm}(k)\sigma$ and $\sigma e^{i\theta\sigma_3}\sigma = -e^{-i\theta\sigma_3}$, we obtain

$$\sigma\varphi_{\pm}^{*}(k^{*})\sigma \sim -Y_{\pm}(k)e^{-i\theta\sigma_3},\ x \to \pm\infty, \tag{2.45}$$

which leads to (2.40).

For the scattering matrix, conjugating on the both sides of Eq. (2.15) leads to

$$\varphi_{+}^{*}(k^{*}) = \varphi_{-}^{*}(k^{*})S^{*}(k^{*}),$$

substituting (2.40) into the above formula yields

$$\varphi_{+}(k) = -\varphi_{-}(k)\sigma S^{*}(k^{*})\sigma,$$

comparing with (2.15) gives

$$S^{*}(k^{*}) = -\sigma S(k)\sigma.$$

Elementwise, Eqs. (2.40)–(2.43) read as

$$\begin{aligned} s_{11}^{*}(k^{*}) &= s_{22}(k), \quad s_{21}^{*}(k^{*}) = -s_{12}(k), \\ s_{11}^{*}(-k^{*}) &= s_{22}(k), \quad s_{21}^{*}(-k^{*}) = s_{12}(k). \end{aligned} \tag{2.46}$$

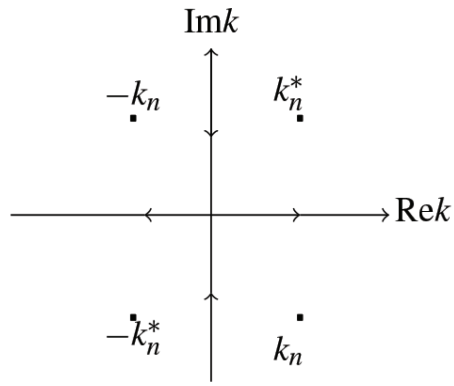

Figure 2 Discrete spectrum.

$$\begin{aligned} &\varphi_{\pm,1}(k)=\sigma\varphi^*_{\pm,2}(k^*), && \varphi_{\pm,2}(k)=-\sigma\varphi^*_{\pm,1}(k^*),\\ &\varphi_{\pm,1}(k)=\sigma_1\varphi^*_{\pm,2}(-k^*), && \varphi_{\pm,2}(k)=\sigma_1\varphi^*_{\pm,1}(-k^*), \end{aligned} \tag{2.47}$$

which implies

$$\varphi_{\pm,2}(k)=-\sigma_3\varphi_{\pm,2}(-k), \qquad \varphi_{\pm,1}(k)=\sigma_3\varphi_{\pm,1}(-k), \tag{2.48}$$

note the relation (2.12) and (2.26), there exists

$$\varphi_{\pm}=e^{i\nu\sigma_3}\mu_{\pm}e^{-i\theta\sigma_3}, \tag{2.49}$$

thus for the eigenfunctions $\mu_{\pm}$, we derive

$$\mu_{\pm,2}(k)=-\sigma_3\mu_{\pm,2}(-k), \qquad \mu_{\pm,1}(k)=\sigma\mu_{\pm,1}(-k). \tag{2.50}$$

3. THE INVERSE SCATTERING WITH NZBCs

3.1. Generalized Riemann–Hilbert Problem

We define the two matrices

$$M^+=\left(\mu_{-,1},\ \frac{\mu_{+,2}}{s_{22}}\right),\ k\in D^+,$$
$$M^-=\left(\frac{\mu_{+,1}}{s_{11}},\ \mu_{-,2}\right),\ k\in D^-,$$

which are analytical in D^+, D^- respectively, and admit asymptotic

$$\begin{aligned} &M^{\pm}=I+O(1/k), && k\to\infty,\\ &M^{\pm}=\frac{1}{k}e^{-i\nu\sigma_3}\widetilde{Q}_-+O(1), && k\to 0, \end{aligned} \tag{3.1}$$

where $\widetilde{Q}_-=\begin{pmatrix} 0 & -\dfrac{1}{q_-^*}\\ \dfrac{1}{q_-} & 0\end{pmatrix}$.

By using (2.49) and (3.1), we rewrite (2.16) and get a generalized Riemann–Hilbert problem

$$\star\ M(x,t,k) \text{ is meromorphic in } \mathbb{C}\setminus\Sigma, \tag{3.2}$$

$$\star\ M^+(x,t,k)=M^-(x,t,k)(I-G(x,t,k)), \qquad k\in\Sigma, \tag{3.3}$$

$$\star\ M(x,t,k) \text{ satisfies residue conditions at zeros } \{k: s_{11}(k)=s_{22}(k)=0\}, \tag{3.4}$$

$$\star\ M^{\pm} = I + O(1/k), \qquad k \to \infty, \tag{3.5}$$

$$\star\ M^{\pm} = \frac{1}{k} e^{-iv\sigma_3} \widetilde{Q}_{\pm} + O(1), \quad k \to 0, \tag{3.6}$$

where the jump matrix

$$G = \begin{pmatrix} 0 & -e^{-2i\theta} \widetilde{\rho}(k) \\ e^{2i\theta} \rho(k) & \rho(k)\widetilde{\rho}(k) \end{pmatrix}, \ \rho(k) = \frac{s_{21}}{s_{11}}, \ \widetilde{\rho}(k) = \frac{s_{12}}{s_{22}}.$$

and $\Sigma = \mathrm{R} \cup i\mathrm{R}$ denotes the jump contour in Figure 1.

3.2. Discrete Spectrum and Residue Conditions

Suppose that $s_{11}(k)$ has N simple zeroes in $D^{+} \cap \{\mathrm{Im}\, k < 0\}$ denoted by k_n, $n = 1, 2, ..., N$. Owing to the symmetries in (2.46), there exists

$$s_{11}(k_n) = 0 \Leftrightarrow s_{22}(-k_n^*) = 0 \Leftrightarrow s_{22}(k_n^*) = 0 \Leftrightarrow s_{11}(-k_n) = 0,$$

thus the discrete spectrum is the set

$$\{\pm k_n, \pm k_n^*\}, \tag{3.7}$$

which distribute in the k-plane as shown in Figure 2.

Next we derive the residue conditions will be needed for the RH problem. If $s_{11}(k) = 0$ at $k = k_n$ the eigenfunctions $\varphi_{+,1}$ and $\varphi_{-,2}$ at $k = k_n$ must be proportional

$$\varphi_{+,1}(k_n) = b_n \varphi_{-,2}(k_n), \tag{3.8}$$

where b_n is an arbitrary constant independent on x, t. Under the transformation (2.49), there exists linear relation for $\mu_{\pm}$

$$\mu_{+,1}(k_n) = b_n e^{2i\theta(k_n)} \mu_{-,2}(k_n). \tag{3.9}$$

Thus we get

$$\mathrm{Res}_{k=k_n}\left[\frac{\mu_{+,1}}{s_{11}}\right] = \frac{\mu_{+,1}(k_n)}{s_{11}'(k_n)} = C_n e^{2i\theta(k_n)} \mu_{-,2}(k_n), \tag{3.10}$$

where $C_n = \dfrac{b_n}{s_{11}'(k_n)}$. As for $k = -k_n$, substituting (2.48) into (3.8) leads to

$$\varphi_{+,1}(-k_n) = -b_n \varphi_{-,2}(-k_n), \tag{3.11}$$

then applying the relation (2.49) yields

$$\mu_{+,1}(-k_n) = -b_n e^{2i\theta(k_n)} \mu_{-,2}(-k_n), \tag{3.12}$$

thus we obtain

$$\mathrm{Res}_{k=-k_n}\left[\frac{\mu_{+,1}}{s_{11}}\right] = \frac{\mu_{+,1}(-k_n)}{s_{11}'(-k_n)} = -C_n e^{2i\theta(k_n)} \sigma_3 \mu_{-,2}(k_n). \tag{3.13}$$

Similarly, the residue conditions at $k = \pm k_n^*$ are

$$\mathrm{Res}_{k=k_n^*}\left[\frac{\mu_{+,2}}{s_{22}}\right] = \frac{\mu_{+,2}(k_n^*)}{s_{22}'(k_n^*)} = \widetilde{C}_n e^{-2i\theta(k_n^*)} \mu_{-,1}(k_n^*), \tag{3.14}$$

$$\mathrm{Res}_{k=-k_n^*}\left[\frac{\mu_{+,2}}{s_{22}}\right] = \frac{\mu_{+,2}(-k_n^*)}{s_{22}'(-k_n^*)} = \widetilde{C}_n e^{-2i\theta(k_n^*)} \sigma_3 \mu_{-,1}(k_n^*), \tag{3.15}$$

where $\widetilde{C}_n = -C_n^*$.

Recall the definition of $M^{\pm}$, there follows

$$\begin{aligned}
&\operatorname{Res}_{k=k_n^*} M^+ = \left(0, \widetilde{C}_n e^{-2i\theta(k_n^*)}\mu_{-,1}(k_n^*)\right), \quad \operatorname{Res}_{k=-k_n^*} M^+ = \left(0, \widetilde{C}_n e^{-2i\theta(k_n^*)}\sigma_3\mu_{-,1}(k_n^*)\right),\\
&\operatorname{Res}_{k=k_n} M^- = \left(C_n e^{2i\theta(k_n)}\mu_{-,2}(k_n), 0\right), \quad \operatorname{Res}_{k=-k_n} M^- = \left(-C_n e^{2i\theta(k_n)}\sigma_3\mu_{-,2}(k_n), 0\right).
\end{aligned} \tag{3.16}$$

3.3. Reconstruction Formula for the Potential

To solve the Riemann–Hilbert problem (3.3), one needs to regularize it by subtracting out the asymptotic behaviour and the pole contribution. Hence, we rewrite Eq. (3.3) as

$$\begin{aligned}
&M^+ - I - \frac{1}{k}e^{-iv\sigma_3}\widetilde{Q}_- - \sum_{n=1}^{N}\frac{\operatorname{Res}_{k=k_n^*}M^+}{k-k_n^*} - \sum_{n=1}^{N}\frac{\operatorname{Res}_{k=k_n}M^-}{k-k_n} - \sum_{n=1}^{N}\frac{\operatorname{Res}_{k=-k_n^*}M^+}{k+k_n^*} - \sum_{n=1}^{N}\frac{\operatorname{Res}_{k=-k_n}M^-}{k+k_n}\\
&= M^- - I - \frac{1}{k}e^{-iv\sigma_3}\widetilde{Q}_- - \sum_{n=1}^{N}\frac{\operatorname{Res}_{k=k_n^*}M^+}{k-k_n^*} - \sum_{n=1}^{N}\frac{\operatorname{Res}_{k=k_n}M^-}{k-k_n} - \sum_{n=1}^{N}\frac{\operatorname{Res}_{k=-k_n^*}M^+}{k+k_n^*} - \sum_{n=1}^{N}\frac{\operatorname{Res}_{k=-k_n}M^-}{k+k_n} - M^-G,
\end{aligned} \tag{3.17}$$

where $\widetilde{Q}_- = \begin{pmatrix} 0 & -\dfrac{1}{q_-^*} \\ \dfrac{1}{q_-} & 0 \end{pmatrix}$. Then the Plemelj's formula shows

$$M(x,t,k) = I + \frac{1}{k}e^{-iv\sigma_3}\widetilde{Q}_- + \sum_{n=1}^{N}\frac{\operatorname{Res}_{k=k_n^*}M^+}{k-k_n^*} + \sum_{n=1}^{N}\frac{\operatorname{Res}_{k=k_n}M^-}{k-k_n} + \sum_{n=1}^{N}\frac{\operatorname{Res}_{k=-k_n^*}M^+}{k+k_n^*} + \sum_{n=1}^{N}\frac{\operatorname{Res}_{k=-k_n}M^-}{k+k_n} + \frac{1}{2\pi i}\int_{\Sigma}\frac{M^-G(x,t,\xi)}{\xi-k}d\xi, \tag{3.18}$$

and the (1, 2)-element of M is

$$M_{12} = \frac{1}{k}\left\{e^{-iv\sigma_3}\widetilde{Q}_- + \sum_{n=1}^{N}(\operatorname{Res}_{k=k_n^*}M^+ + \operatorname{Res}_{k=-k_n^*}M^+) - \frac{1}{2\pi i}\int_{\Sigma}\frac{M^-G(x,t,\xi)}{\xi-k}d\xi\right\}_{12} + O(1/k^2). \tag{3.19}$$

Comparing the (1, 2)-element on the both sides of Eq. (2.20) yields

$$q_x - i\beta q = 2ie^{iv}J^{(1)}_{\pm,12}. \tag{3.20}$$

Recall the transformation (2.2), we know

$$u_x e^{i\tau(x,t)} = q_x - i\beta q,$$

where $\tau(x,t) = \beta x + \alpha\beta^2 q_0^2 t$. Thus we can write (3.20) as

$$u_x = 2ie^{iv-i\tau}\lim_{k\to\infty}(kJ_{\pm})_{12} = 2ie^{2iv-i\tau}\lim_{k\to\infty}(k\mu_{\pm})_{12} = 2ie^{2iv-i\tau}\left(M_1^-\right)_{12}, \tag{3.21}$$

where

$$M^- = M_0^- + \frac{M_1^-}{k} + \cdots.$$

Substituting (3.19) and (3.16) into (3.21), we obtain the reconstruction formula for potential

$$u_x = 2ie^{2iv-i\tau}\left\{\frac{e^{-iv}}{q_-^*} + 2\sum_{n=1}^{N}\widetilde{C}_n e^{-2i\theta(k_n^*)}\mu_{-,1,1}(k_n^*) - \frac{1}{2\pi i}\int_{\Sigma}(M^-G)_{12}(\xi)d\xi\right\}. \tag{3.22}$$

4. REFLECTIONLESS POTENTIALS

Now we consider the potential $u(x, t)$ for which the reflection coefficient $\rho(k)$ vanishes identically, that is, $G = 0$. In this case, Eq. (3.22) reads as

$$u_x = 2ie^{-i\tau}\left(\sum_n 2e^{2iv}\widetilde{C}_n e^{-2i\theta(k_n^*)}\mu_{-,1,1}(k_n^*) - \frac{e^{iv}}{q_-^*}\right). \tag{4.1}$$

To obtain the expression of the term $\mu_{-,1,1}(k_n^*)$, we consider the first and second column of (3.18) respectively under reflectionless case:

$$\begin{aligned}\mu_{-,2}(k_n) &= \begin{pmatrix} -\dfrac{e^{-iv}}{k_n q_-^*} \\ 1 \end{pmatrix} + \sum_{j=1}^{N} \frac{\widetilde{C}_j e^{-2i\theta(k_j^*)}}{k_n - k_j^*} \mu_{-,1}(k_j^*) + \sum_{j=1}^{N} \frac{\widetilde{C}_j e^{-2i\theta(k_j^*)}}{k_n + k_j^*} \sigma_3 \mu_{-,1}(k_j^*), \\ \mu_{-,1}(k_n^*) &= \begin{pmatrix} 1 \\ \dfrac{e^{iv}}{k_n^* q_-} \end{pmatrix} + \sum_{j=1}^{N} \frac{C_j e^{2i\theta(k_j)}}{k_n^* - k_j} \mu_{-,2}(k_j) - \sum_{j=1}^{N} \frac{C_j e^{2i\theta(k_j)}}{k_n^* + k_j} \sigma_3 \mu_{-,2}(k_j),\end{aligned} \tag{4.2}$$

which can be further written as

$$\begin{aligned}\mu_{-,2}(k_n) &= \begin{pmatrix} -\dfrac{e^{-iv}}{k_n q_-^*} \\ 1 \end{pmatrix} + 2\sum_{j=1}^{N} \frac{\widetilde{C}_j e^{-2i\theta(k_j^*)}}{k_n^2 - (k_j^*)^2} K_1 \mu_{-,1}(k_j^*), \\ \mu_{-,1}(k_n^*) &= \begin{pmatrix} 1 \\ \dfrac{e^{iv}}{k_n^* q_-} \end{pmatrix} + 2\sum_{j=1}^{N} \frac{C_j e^{2i\theta(k_j)}}{(k_n^*)^2 - k_j^2} K_2 \mu_{-,2}(k_j),\end{aligned} \tag{4.3}$$

where

$$K_1 = \begin{pmatrix} k_n & 0 \\ 0 & k_j^* \end{pmatrix}, \quad K_2 = \begin{pmatrix} k_j & 0 \\ 0 & k_n^* \end{pmatrix}.$$

Define

$$c_j(x,t,k) = \frac{C_j}{(k^*)^2 - k_j^2} e^{2i\theta(x,t,k_j)},$$

whose conjugate gives

$$c_j^*(k) = -\frac{\widetilde{C}_j}{k^2 - (k_j^*)^2} e^{-2i\theta(k_j^*)}.$$

Then (4.3) reduces to

$$\mu_{-,1,1}(k_n^*) = 1 + 2\sum_{j=1}^{N} k_j c_j(k_n)\, \mu_{-,1,2}(k_j), \tag{4.4}$$

$$\mu_{-,1,2}(k_j) = -\frac{e^{-iv}}{k_j q_-^*} - 2\sum_{m=1}^{N} k_j c_m^*(k_j)\, \mu_{-,1,1}(k_m^*), \tag{4.5}$$

substituting (4.5) into (4.4) yields

$$\mu_{-,1,1}(k_n^*) = 1 - \frac{2e^{-iv}}{q_-^*} \sum_{j=1}^{N} c_j(k_n) - 4\sum_{j=1}^{N}\sum_{m=1}^{N} k_j^2 c_j(k_n) c_m^*(k_j) \mu_{-,1,1}(k_m^*), \quad n = 1, 2, \dots, N. \tag{4.6}$$

Introducing notations

$$X = (X_1, X_2, \dots, X_N)^t, \qquad A = (A_{n,m}), \qquad B = (B_1, B_2, \dots, B_N)^t$$

with components being

$$X_n = \mu_{-,1,1}(k_n^*), \qquad A_{nm} = \sum_{j=1}^{N} 4k_j^2 c_j(k_n) c_m^*(k_j), \qquad B_n = 1 - \frac{2e^{-iv}}{q_-^*} \sum_{j=1}^{N} c_j(k_n),$$

then the system (4.6) can be written as matrix form

$$HX = B, \tag{4.7}$$

where

$$H = I + A = (H_1, H_2, \dots, H_N).$$

By standard Crammer rule, the system (4.7) is the solution of

$$\mu_{-,1,1}(k_n^*) = X_n = \frac{\det H_n^{\text{ext}}}{\det H}, \tag{4.8}$$

where

$$H_n^{\text{ext}} = (H_1, \ldots, H_{n-1}, B, \ldots, H_N).$$

Note that

$$u_x v_x = q_x r_x + i\beta q_x r - i\beta q r_x + \beta^2 q r,$$

then Eq. (2.24) reduces to

$$\nu = \int_{-\infty}^{x} (\beta + \frac{1}{2} u_x v_x)(x', t)\, dx', \tag{4.9}$$

Therefore, we obtain a compact solution:

$$u_x = 2ie^{-i\tau(x,t)+i\nu}\left(\frac{\det H^{\text{aug}}}{\det H} e^{i\nu} - \frac{1}{q_-^*}\right), \tag{4.10}$$

where the augmented $(N+1)\times(N+1)$ matrix H^{aug} is

$$H^{\text{aug}} = \begin{pmatrix} 0 & Y^t \\ B & H \end{pmatrix}, \quad Y = (Y_1, \ldots, Y_N)^t,$$

and $Y_n = 2\widetilde{C}_n e^{-2i\theta(k_n^*)} = -2C_n^* e^{-2i\theta(k_n^*)}$.

5. TRACE FORMULA AND THETA CONDITION

Define

$$\beta^- = s_{11}(k)\prod_{n=1}^{N} \frac{k^2 - (k_n^*)^2}{k^2 - k_n^2}, \quad \beta^+ = s_{22}(k)\prod_{n=1}^{N} \frac{k^2 - k_n^2}{k^2 - (k_n^*)^2}, \tag{5.1}$$

we see that they are analytic and no-zeros in D^- and D^+, respectively. Moreover, $\beta^+\beta^- = s_{11}(k)s_{22}(k)$. Note that $\det S(k) = s_{11}s_{22} - s_{21}s_{12} = 1$, this implies

$$\frac{1}{s_{11}s_{22}} = 1 - \rho(k)\,\widetilde{\rho}(k) = 1 + \rho(k)\,\rho^*(k^*),$$

thus

$$\beta^+\beta^- = s_{11}s_{22} = \frac{1}{1+\rho(k)\,\rho^*(k^*)},\ k \in \Sigma.$$

Taking logarithms leads to

$$\log\beta^+ - (-\log\beta^-) = -\log[1+\rho(k)\,\rho(k^*)],\ k\in\Sigma,$$

then Applying Plemelj formula, we have

$$\log\beta^{\pm} = \mp\frac{1}{2\pi i}\int_{\Sigma}\frac{\log[1+\rho(s)\,\rho^*(s^*)]}{s-k}\,ds,\ k\in D^{\pm}. \tag{5.2}$$

Substituting into (5.1), we obtain the trace formula

$$\begin{aligned} s_{11}(k) &= \exp\left[\frac{1}{2\pi i}\int_{\Sigma}\frac{\log[1+\rho(s)\,\rho^*(s^*)]}{s-k}\,ds\right]\prod_{n=1}^{N}\frac{k^2-k_n^2}{k^2-(k_n^*)^2}, \qquad k\in D^-, \\ s_{22}(k) &= \exp\left[-\frac{1}{2\pi i}\int_{\Sigma}\frac{\log[1+\rho(s)\rho^*(s^*)]}{s-k}\,ds\right]\prod_{n=1}^{N}\frac{k^2-(k_n^*)^2}{k^2-k_n^2}, \qquad k\in D^+. \end{aligned} \tag{5.3}$$

Under reflectionless condition, they reduce to

$$s_{11}(k)=\prod_{n=1}^{N}\frac{k^2-k_n^2}{k^2-(k_n^*)^2},\ k\in D^-;\qquad s_{22}(k)=\prod_{n=1}^{N}\frac{k^2-(k_n^*)^2}{k^2-k_n^2},\ k\in D^+. \tag{5.4}$$

Taking limit as $k\to 0$ for (5.4) leads to

$$\frac{q_-}{\beta q^+}=\exp\left[-\frac{i}{2\pi}\int_\Sigma\frac{\log[1+\rho(s)\,\rho^*(s^*)]}{s}\,ds\right]\exp\left[4i\sum_{n=1}^{N}\arg(k_n)\right],\ k\in D^-, \tag{5.5}$$

note that β is a positive constant, then we obtain the theta condition

$$\arg\left(\frac{q_-}{q_+}\right)=-\frac{1}{2\pi}\int_\Sigma\frac{\log[1+\rho(s)\rho^*(s^*)]}{s}ds+4\sum_{n=1}^{N}\arg(k_n). \tag{5.6}$$

under reflectionless condition, we have

$$\arg\left(\frac{q_-}{q_+}\right)=4\sum_{n=1}^{N}\arg(k_n). \tag{5.7}$$

6. ONE-SOLITON SOLUTION

As an application of the formula (4.10) of N-soliton solution, we construct one-soliton solution for the FL equation, which corresponds to $N=1$. Then Eq. (4.10) becomes

$$u_x=2ie^{-i\tau+i\nu}\left(-\frac{2C_1^*e^{-2i\theta(k_1^*)}e^{i\nu}}{1+4k_1^2\,|c_1(k_1)|^2}+\frac{(4+4\beta)C_1^*e^{-2i\theta(k_1^*)}c_1(k_1)}{q_-^*(1+4k_1^2\,|c_1(k_1)|^2)}-\frac{1}{q_-^*}\right), \tag{6.1}$$

where k_1 is an eigenvalue, C_1 is an arbitrary constant and

$$\begin{aligned}&\theta(k_1)=\frac{k_1\lambda_1}{\sqrt{\alpha}}x+\lambda_1\eta_1 t, &&\lambda_1=\sqrt{\alpha}\left(k_1+\frac{\beta}{2k_1}\right),\\ &\eta_1=\sqrt{\alpha}\left(k_1-\frac{\beta}{2k_1}\right), &&c_1(k_1)=\frac{C_1}{(k_1^*)^2-k_1^2}e^{2i\theta(k_1)}.\end{aligned} \tag{6.2}$$

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

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