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On Moments Properties of Generalized Order Statistics from Marshall-Olkin-Extended General Class of Distribution

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ABSTRACT

Marshall and Olkin [Biometrika. 84 (1997), 641–652. https://doi.org/10.1093/biomet/84.3.641] introduced a new method of adding parameter to expand a family of distribution. In this paper the Marshall-Olkin extended general class of distribution is used. Further, some recurrence relations for single and product moments of generalized order statistics (gos) are studied. Also the results are deduced for order statistics and record values.

1. INTRODUCTION

Kamps [2] introduced the unifying concept of generalized order statistics (gos), the use of such concept has been steadily growing along the years. This is due to the fact that such concept describes random variables arranged in ascending order of magnitude and includes important well known concept that have been separately treated in statistical literature. Examples of such concepts are the order statistics, sequential order statistics, progressive type II censored order statistics, record values and pfeifer's records. Application is multifarious in a variety of disciplines and particularly in reliability.

Let $n \geq 2$ be a given integer and $\widetilde{m} = (m_1, m_2, \dots, m_{n-1}) \in \Re^{n-1}$, $k \geq 1$ be the parameters such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j \geq 0 \ \ for \ \ 1 \leq i \leq n-1.$$

The random variables $X(1, n, \widetilde{m}, k), X(2, n, \widetilde{m}, k), \dots, X(n, n, \widetilde{m}, k)$ are said to be generalized order statistics from an absolutely continuous distribution function $F()$ with the probability density funtion (pdf) $f()$, if their joint density function is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n), \tag{1}$$

on the cone $F^{-1}(0) < x_1 \leq x_2 \leq \dots \leq x_n < F^{-1}(1)$.

If $m_i = 0$, $i = 1, 2, \dots, n-1$ and $k = 1$, we obtain the joint *pdf* of the order statistics and for $m_i = -1$, $k \in N$, we get the joint *pdf* $k-th$ record values.

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Let the Marshall-Olkin extended general form of distribution be

$$\overline{F}(x) = \frac{\lambda\, e^{-\frac{h(x)}{c}}}{\left[1-(1-\lambda)\, e^{-\frac{h(x)}{c}}\right]}, \quad \alpha \le x \le \beta,\ \lambda > 0, \tag{2}$$

where c is such that $F(\alpha)=0,\ F(\beta)=1$ and $h(x)$ is a monotonic and differentiable function of x in the interval (α,β).

Also we have,

$$\overline{F}(x) = \frac{c}{h'(x)}\left[1-(1-\lambda)\, e^{-\frac{h(x)}{c}}\right] f(x), \tag{3}$$

where, $\overline{F}(x) = 1-F(x)$. The relation (3) will be utilized to establish recurrence relations for moments of *gos*.

2. RELATIONS FOR SINGLE MOMENTS

Case I: $\gamma_i \neq \gamma_j;\ i \neq j = 1,2,\ldots,n-1.$

In view of (1) the *pdf* of $r-th$ generalized order statistic $X(r,n,\widetilde{m},k)$ is

$$f_{X(r,n,\widetilde{m},k)}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r) \left[\overline{F}(x)\right]^{\gamma_i - 1}, \tag{4}$$

where,

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i,\ \gamma_i = k+n-i+\sum_{j=i}^{n-1} m_j > 0,$$

and

$$a_i(r) = \prod_{\substack{j=1 \\ j\neq i}}^{r} \frac{1}{(\gamma_j-\gamma_i)}, 1 \le i \le r \le n.$$

Theorem 2.1. For the Marshall-Olkin extended general class of distributions as given in (2) and $n \in N,\ \widetilde{m} \in \Re,\ k>0, 1 \le r \le n, \lambda > 0$

$$\begin{aligned} E\left[\xi\left\{X(r,n,\widetilde{m},k)\right\}\right] &= E\left[\xi\left\{X(r-1,n,\widetilde{m},k)\right\}\right] + \frac{c}{\gamma_r} E\left[\phi\left\{X(r,n,\widetilde{m},k)\right\}\right] \\ &\quad - \frac{c\,(1-\lambda)}{\gamma_r} E\left[\psi\left\{X(r,n,\widetilde{m},k)\right\}\right], \end{aligned} \tag{5}$$

where $\phi(x) = \dfrac{\xi'(x)}{h'(x)}$ and $\psi(x) = \dfrac{\xi'(x)}{h'(x)} e^{-\frac{h(x)}{c}}$.

Proof: We have by Athar and Islam [3],

$$\begin{aligned} &E\left[\xi\left\{X(r,n,\widetilde{m},k)\right\}\right] - E\left[\xi\left\{X(r-1,n,\widetilde{m},k)\right\}\right] \\ &= C_{r-2} \int_{\alpha}^{\beta} \xi'(x) \sum_{i=1}^{r} a_i(r) \left[\overline{F}(x)\right]^{\gamma_i} dx. \end{aligned} \tag{6}$$

Now on using (3) in (6), we get

$$\begin{aligned} &E\left[\xi\left\{X(r,n,\widetilde{m},k)\right\}\right] - E\left[\xi\left\{X(r-1,n,\widetilde{m},k)\right\}\right] \\ &= \frac{c\, C_{r-1}}{\gamma_r} \int_{\alpha}^{\beta} \frac{\xi'(x)}{h'(x)} \sum_{i=1}^{r} a_i(r) [\overline{F}(x)]^{\gamma_i - 1} \left[1-(1-\lambda)\, e^{-\frac{h(x)}{c}}\right] f(x)\, dx. \end{aligned}$$

which after simplification yields (5).

Case II: $m_i = m,\ i = 1, 2, \ldots, n-1$.

The *pdf* of $X(r, n, m, k)$ is given as

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}\left[\bar{F}(x)\right]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x)), \tag{7}$$

where,

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i,\ \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \neq -1. \\ -\log(1-x), & m = -1. \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0),\ x \in (0,1).$$

Theorem 2.2. For the Marshall-Olkin extended general class of distributions as given in (2) and $n \in N$, $\widetilde{m} \in \Re$, $k > 0, 1 \le r \le n, \lambda > 0$

$$\begin{aligned} E[\xi\{X(r,n,m,k)\}] &= E[\xi\{X(r-1,n,m,k)\}] + \frac{c}{\gamma_r} E[\phi\{X(r,n,m,k)\}] \\ &\quad - \frac{c(1-\lambda)}{\gamma_r} E[\psi\{X(r,n,m,k)\}]. \end{aligned} \tag{8}$$

Proof: It may be noted that for $\gamma_i \neq \gamma_j$ but at $m_i = m,\ i = 1, 2, \ldots, n-1$,

$$a_i(r) = \frac{1}{(m+1)^{r-1}}(-1)^{r-i}\frac{1}{(i-1)!(r-i)!}.$$

Therefore the *pdf* of $X(r, n, \widetilde{m}, k)$ given in (4) reduces to (7) (*cf* Khan *et al.* [4]).

Hence it can be seen that (8) is the partial case of (5) and is obtained by replacing $\widetilde{m}$ with m in (5).

Remark 2.1. Recurrence relation for single moments of order statistics (at $m = 0, k = 1$) is

$$E[\xi(X_{r:n})] = E[\xi(X_{r-1:n})] + \frac{c}{(n-r+1)}\{E[\phi(X_{r:n})] - (1-\lambda)\ E[\psi(X_{r:n})]\},$$

at $\lambda = 1$, we get

$$E[\xi(X_{r:n})] = E[\xi(X_{r-1:n})] + \frac{c}{(n-r+1)} E[\phi(X_{r:n})],$$

as obtained by Ali and Khan [5].

Remark 2.2. Recurrence relation for single moments of $k-th$ upper record (at $m = -1$) will be

$$\begin{aligned} E[\xi\{X(r,n,-1,k)\}] &= E[\xi\{X(r-1,n,-1,k)\}] + \frac{c}{k}\{E[\phi\{X(r,n,-1,k)\}] \\ &\quad - (1-\lambda)\ E[\psi\{X(r,n,-1,k)\}]\}. \end{aligned}$$

Remark 2.3. Setting $\lambda = 1$ in (8), we get

$$E[\xi\{X(r,n,m,k)\}] = E[\xi\{X(r-1,n,m,k)\}] + \frac{c}{\gamma_r} E[\phi\{X(r,n,m,k)\}],$$

as obtained by Anwar *et al.* [6].

EXAMPLES

i. **Marshall-Olkin-Extended Exponential Distribution**

$$\overline{F}(x) = \frac{\lambda\, e^{-\theta x}}{\left[1 + (1-\lambda)\, e^{-\theta x}\right]}, \quad 0 < x < \infty,\ \lambda > 0,$$

we have,

$$c = \frac{1}{\theta} \text{ and } h(x) = x,$$

let $\xi(x) = x^{j+1}$, then

$$\phi(x) = (j+1)\, x^j \text{ and } \psi(x) = (j+1) \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \theta^p\, x^{j+p}.$$

Thus from relation (8), we have

$$\begin{aligned} E\left[X^{j+1}(r,n,m,k)\right] - E\left[X^{j+1}(r-1,n,m,k)\right] = \frac{(j+1)}{\theta\, \gamma_r} \Bigg\{ & E\left[X^j(r,n,m,k)\right] \\ & - (1-\lambda) \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \theta^p\, E\left[X^{j+p}(r,n,m,k)\right] \Bigg\}. \end{aligned} \tag{9}$$

ii. **Marshall-Olkin-Extended Erlang Truncated Exponential Distribution**

$$\overline{F}(x) = \frac{\lambda\, e^{-\alpha\left(1-e^{-\beta}\right)x}}{\left[1 - (1-\lambda)\, e^{-\alpha\left(1-e^{-\beta}\right)x}\right]}, \quad 0 < x < \infty,\ \lambda > 0,\ \alpha,\ \beta > 0,$$

here we have

$$c = \frac{1}{\alpha\left(1-e^{-\beta}\right)} \text{ and } h(x) = x,$$

assuming $\xi(x) = x^{j+1}$, we get

$$\phi(x) = (j+1)\, x^j \text{ and } \psi(x) = (j+1) \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \left[\alpha\left(1-e^{-\beta}\right)\right]^p x^{j+p}.$$

Thus from relation (8),

$$\begin{aligned} E\left[X^{j+1}(r,n,m,k)\right] = E\left[X^{j+1}(r-1,n,m,k)\right] + \frac{(j+1)}{\gamma_r\left[\alpha\left(1-e^{-\beta}\right)\right]} \Bigg\{ & E\left[X^j(r,n,m,k)\right] \\ & - (1-\lambda) \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \left[\alpha\left(1-e^{-\beta}\right)\right]^p E\left[X^{j+p}(r,n,m,k)\right] \Bigg\}. \end{aligned} \tag{10}$$

iii. **Marshall-Olkin-Extended Rayleigh Distribution**

$$\overline{F}(x) = \frac{\lambda\, e^{-\frac{x^2}{2\theta^2}}}{\left[1 + (1-\lambda)\, e^{-\frac{x^2}{2\theta^2}}\right]}, \quad 0 < x < \infty,\ \lambda > 0,\ \theta > 0.$$

we have,

$$c = 2\theta^2 \text{ and } h(x) = x^2,$$

let $\xi(x) = x^{j+1}$, then

$$\phi(x) = \frac{(j+1)}{2} x^{j-1} \text{ and } \psi(x) = (j+1) \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \frac{1}{2^{p+1}} \frac{1}{\theta^{2p}} x^{2p+j-1}.$$

Thus from relation (8), we have

$$E\left[X^{j+1}(r,n,m,k)\right]-E\left[X^{j+1}(r-1,n,m,k)\right]=\frac{\theta^2(j+1)}{\gamma_r}\Bigg\{E\left[X^{j-1}(r,n,m,k)\right]-(1-\lambda)\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\frac{1}{(2\theta^2)^p}E\left[X^{j+2p-1}(r,n,m,k)\right]\Bigg\}. \tag{11}$$

3. RELATIONS FOR PRODUCT MOMENTS

Case I: $\gamma_i \neq \gamma_j;\ i \neq j = 1, 2, \ldots, n-1.$

The joint probability density function *pdf* of $X(r,n,\widetilde{m},k)$ and $X(s,n,\widetilde{m},k)$, $1 \leq r < s \leq n$ is given as

$$f_{X(r,n,\widetilde{m},k),X(s,n,\widetilde{m},k)}(x,y)=C_{s-1}\left(\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_i}\right)\left(\sum_{i=1}^{r}a_i(r)\left[\overline{F}(x)\right]^{\gamma_i}\right)\times\left(\sum_{i=1}^{r}a_i(r)\left[\overline{F}(x)\right]^{\gamma_i}\right)\frac{f(x)}{\overline{F}(x)}\frac{f(y)}{\overline{F}(y)},\quad \alpha\leq x<y\leq\beta, \tag{12}$$

where,

$$a_i^{(r)}(s)=\prod_{\substack{j=r+1\\ j\neq i}}^{s}\frac{1}{\gamma_j-\gamma_i},\quad r+1\leq i\leq s\leq n.$$

Theorem 3.1. For the Marshall-Olkin extended general class of distributions as given in (2). Fix a positive integer k and for $n \in N$, $\widetilde{m} \in \mathfrak{R}$, $1 \leq r < s \leq n$,

$$\begin{aligned}E[\xi\{X(r,n,\widetilde{m},k),X(s,n,\widetilde{m},k)\}]&=E[\xi\{X(r,n,\widetilde{m},k),X(s-1,n,\widetilde{m},k)\}]\\&+\frac{c}{\gamma_s}E[\phi\{X(r,n,\widetilde{m},k),X(s,n,\widetilde{m},k)\}]\\&-\frac{c(1-\lambda)}{\gamma_s}E[\psi\{X(r,n,\widetilde{m},k),X(s,n,\widetilde{m},k)\}],\end{aligned} \tag{13}$$

where,

$$\phi(x,y)=\frac{\frac{\partial}{\partial y}\xi(x,y)}{h'(y)},\quad \psi(x,y)=e^{-\frac{h(y)}{c}}\frac{\frac{\partial}{\partial y}\xi(x,y)}{h'(y)},\quad \xi(x,y)=\xi_1(x).\xi_2(y).$$

Proof: We have by Athar and Islam [3],

$$\begin{aligned}&E[\xi\{X(r,n,\widetilde{m},k),X(s,n,\widetilde{m},k)\}]-E[\xi\{X(r,n,\widetilde{m},k),X(s-1,n,\widetilde{m},k)\}]\\&=C_{s-2}\int\int_{\alpha\leq x<y\leq\beta}\frac{\partial}{\partial y}\xi(x,y)\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_i}\sum_{i=1}^{r}a_i(r)\left[\overline{F}(x)\right]^{\gamma_i}\frac{f(x)}{\overline{F}(x)}\,dy\,dx.\end{aligned} \tag{14}$$

Now in view of (3) and (14), we have

$$\begin{aligned}&E[\xi\{X(r,n,\widetilde{m},k),X(s,n,\widetilde{m},k)\}]-E[\xi\{X(r,n,\widetilde{m},k),X(s-1,n,\widetilde{m},k)\}]\\&=\frac{c}{\gamma_s}C_{s-1}\int\int_{\alpha\leq x<y\leq\beta}\frac{\frac{\partial}{\partial y}\xi(x,y)}{h'(y)}\left(\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_i}\right)\left(\sum_{i=1}^{r}a_i(r)\left[\overline{F}(x)\right]^{\gamma_i}\right)\\&\times\left\{\left[1-(1-\lambda)e^{-\frac{h(y)}{c}}\right]\right\}\frac{f(x)}{\overline{F}(x)}\frac{f(y)}{\overline{F}(y)}\,dy\,dx,\end{aligned} \tag{15}$$

which leads to (13).

Case II: $m_i = m;\ i = 1, 2, \ldots, n-1.$

The joint *pdf* of $X(r, n, m, k)$ and $X(s, n, m, k)$, $1 \le r < s \le n$ is given as

$$f_{X(r,n,m,k),X(s,n,m,k)}(x, y) = \frac{C_{s-1}}{(r-1)!\,(s-r-1)!}\left[\overline{F}(x)\right]^{m} f(x)\, g_m^{r-1}(F(x)) \times \left[h_m(F(y)) - h_m(F(x))\right]^{s-r-1}\left[\overline{F}(y)\right]^{\gamma_s - 1} f(y),\ \alpha \le x < y \le \beta. \quad (16)$$

Theorem 3.2. For distribution as given in (2) and condition stated as in Theorem 3.1.

$$E[\xi\{X(r,n,m,k), X(s,n,m,k)\}] = E[\xi\{X(r,n,m,k), X(s-1,n,m,k)\}] + \frac{c}{\gamma_s} E[\phi\{X(r,n,m,k), X(s,n,m,k)\}] - \frac{c(1-\lambda)}{\gamma_s} E[\psi\{X(r,n,m,k), X(s,n,m,k)\}]. \quad (17)$$

Proof: We have when $\gamma_i \ne \gamma_j$ but at $m_i = m_j = m$,

$$a_i^{(r)}(s) = \frac{1}{(m+1)^{s-r-1}}(-1)^{s-i}\frac{1}{(i-r-1)!\,(s-i)!},$$

hence, joint *pdf* of $X(r, n, \widetilde{m}, k)$ and $X(s, n, \widetilde{m}, k)$ given in (12) reduces to (16). (*cf* Khan *et al.* [4]). Therefore, Theorem 3.2 can be established by replacing $\widetilde{m}$ with m in Theorem 3.1.

Remark 3.1. Recurrence relation for product moments of order statistics (at $m = 0,\ k = 1$) is

$$E\left[\xi\left(X_{r,s:n}\right)\right] = E\left[\xi\left(X_{r,s-1:n}\right)\right] + \frac{c}{(n-s+1)}\left\{E\left[\phi\left(X_{r,s:n}\right)\right] - (1-\lambda)\, E\left[\psi\left(X_{r,s:n}\right)\right]\right\}, \quad (18)$$

at $\lambda = 1$, we get

$$E\left[\xi\left(X_{r,s:n}\right)\right] = E\left[\xi\left(X_{r,s-1:n}\right)\right] + \frac{c}{(n-s+1)}\, E\left[\phi\left(X_{r,s:n}\right)\right], \quad (19)$$

as obtained by Ali and Khan [7].

Remark 3.2. Recurrence relation for product moments of $k-th$ record values will be

$$E[\xi\{X(r,n,-1,k), X(s,n,-1,k)\}] = E[\xi\{X(r,n,-1,k), X(s-1,n,-1,k)\}] + \frac{c}{k}\{E[\phi\{X(r,n,-1,k), X(s,n,-1,k)\}] - (1-\lambda)\, E[\psi\{X(r,n,-1,k), X(s,n,-1,k)\}]\}.$$

Remark 3.3. Set $\lambda = 1$ in (17), we get

$$E[\xi\{X(r,n,m,k), X(s,n,m,k)\}] = E[\xi\{X(r,n,m,k), X(s-1,n,m,k)\}] + \frac{c}{\gamma_s} E[\phi\{X(r,n,m,k), X(s,n,m,k)\}],$$

as obtained by Anwer *et al.* [6].

EXAMPLES

i. **Marshall-Olkin-Extended Exponential Distribution**

$$\overline{F}(x) = \frac{\lambda\, e^{-\theta x}}{\left[1 + (1-\lambda)\, e^{-\theta x}\right]},\quad 0 < x < \infty,\ \lambda > 0,$$

we have,

$$c = \frac{1}{\theta}\ \text{and}\ h(x) = x,$$

let $\xi\left(x,y\right)=x^{i}y^{j+1}$, then

$$\phi\left(x,y\right)=\left(j+1\right)\,x^{i}y^{j}\text{ and }\psi\left(x\right)=\left(j+1\right)\sum_{p=0}^{\infty}\frac{(-1)^{p}}{p!}\theta^{p}\,x^{i}y^{j+p}.$$

Thus from relation (8), we have

$$\begin{aligned}&E\left[X^{i}\left(r,n,m,k\right),X^{j+1}\left(s,n,m,k\right)\right]-E\left[X^{i}\left(r,n,m,k\right),X^{j+1}\left(s-1,n,m,k\right)\right]\\&=\frac{\left(j+1\right)}{\theta\gamma_{s}}\left\{E\left[X^{i}\left(r,n,m,k\right),X^{j}\left(s,n,m,k\right)\right]-(1-\lambda)\sum_{p=0}^{\infty}\frac{(-1)^{p}}{p!}\theta^{p}E\left[X^{i}\left(r,n,m,k\right),X^{j+p}\left(s,n,m,k\right)\right]\right\}.\end{aligned}$$

ii. **Marshall-Olkin-Extended Erlang Truncated Exponential Distribution**

$$\overline{F}(x)=\frac{\lambda\,e^{-\alpha\left(1-e^{-\beta}\right)x}}{\left[1-(1-\lambda)\,e^{-\alpha(1-e^{-\beta})x}\right]},\quad 0<x<\infty,\ \lambda>0,\ \alpha,\ \beta>0.$$

Here we have

$$c=\frac{1}{\alpha\left(1-e^{-\beta}\right)}\text{ and }h\left(x\right)=x,$$

assuming $\xi\left(x,y\right)=x^{i}y^{j+1}$, we get

$$\phi\left(x,y\right)=\left(j+1\right)x^{i}y^{j}\text{ and }\psi\left(x,y\right)=\left(j+1\right)\sum_{p=0}^{\infty}\frac{(-1)^{p}}{p!}\left[\alpha\left(1-e^{-\beta}\right)\right]^{p}x^{i}y^{j+p}.$$

Thus from relation (8),

$$\begin{aligned}E\left[X^{i}\left(r,n,m,k\right),X^{j}\left(s,n,m,k\right)\right]&=E\left[X^{i}\left(r,n,m,k\right),X^{j}\left(s-1,n,m,k\right)\right]\\&+\frac{\left(j+1\right)}{\gamma_{s}\left[\alpha\left(1-e^{-\beta}\right)\right]}\left\{E\left[X^{i}\left(r,n,m,k\right),X^{j}\left(s,n,m,k\right)\right]\right.\\&\left.-\ (1-\lambda)\sum_{p=0}^{\infty}\frac{(-1)^{p}}{p!}\left[\alpha\left(1-e^{-\beta}\right)\right]^{p}E\left[X^{i}\left(r,n,m,k\right),X^{j+p}\left(s,n,m,k\right)\right]\right\}.\end{aligned}\tag{20}$$

iii. **Marshall-Olkin-Extended Rayleigh Distribution**

$$\overline{F}(x)=\frac{\lambda\,e^{-\frac{x^{2}}{2\theta^{2}}}}{\left[1+(1-\lambda)\,e^{-\frac{x^{2}}{2\theta^{2}}}\right]},\quad 0<x<\infty,\ \lambda>0,\ \theta>0.$$

We have,

$$c=2\theta^{2}\text{ and }h\left(x\right)=x^{2},$$

let $\xi\left(x,y\right)=x^{i}y^{j+1}$, then

$$\phi\left(x,y\right)=\frac{\left(j+1\right)}{2}x^{i}y^{j-1}\text{ and }\psi\left(x,y\right)=\left(j+1\right)\sum_{p=0}^{\infty}\frac{(-1)^{p}}{p!}\frac{1}{2^{p+1}}\frac{1}{\theta^{2p}}x^{i}y^{2p+j-1}.$$

Thus from relation (8), we have

$$\begin{aligned}&E\left[X^{i}\left(r,n,m,k\right),X^{j+1}\left(s,n,m,k\right)\right]-E\left[X^{i}\left(r,n,m,k\right),X^{j+1}\left(s-1,n,m,k\right)\right]\\&=\frac{\theta^{2}\left(j+1\right)}{\gamma_{r}}\left\{E\left[X^{i}\left(r,n,m,k\right),X^{j-1}\left(s,n,m,k\right)\right]\right.\\&\left.-(1-\lambda)\sum_{p=0}^{\infty}\frac{(-1)^{p}}{p!}\frac{1}{(2\theta^{2})^{p}}E\left[X^{i}\left(r,n,m,k\right),X^{j+2p-1}\left(s,n,m,k\right)\right]\right\}.\end{aligned}$$

CONFLICT OF INTEREST

The authors declare that they have no competing interests.

AUTHORS' CONTRIBUTIONS

All authors equally contributed in the manuscript. All authors read and approved the final manuscript.

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