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Free Mechanical Vibrations Models via p-fuzzy Systems

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Abstract

In this work, we use a Mamdani fuzzy controller based on fuzzy rules that represent position and velocity of a particle for a free mechanical vibration model. Here, we focus on three major cases: the harmonic oscillator, the damped vibration, and the negative damping case. This study combines classical numerical methods for ordinary differential equations and fuzzy rule-based systems to describe the dynamical interaction between the position and velocity. In this regard, we propose a p-fuzzy system to study this phenomenon. Computational simulations reveal that the solution of the proposed p-fuzzy system is similar to the corresponding analytic solution of the classical free mechanical vibration problems.

Keywords: Free mechanical vibrations, fuzzy numbers, Mamdani fuzzy controller, p-fuzzy systems.

1 Introduction

The study of vibrations is fundamental to comprehend several physical phenomena and to design many structures such as computational design of engines, planes or cars [5]. Considering the vibrations in the projects of such structures helps to reduce costs and predict possible problems related to noise or attrition. Moreover, for many applications are required approximations for the natural frequencies and damping ratios [3].

In general, free mechanical vibrations models are given by differential equations that describes the dynamics of a particle restricted to physical conditions in a mass-spring-damper system. Obtaining analytic solutions to these problems requires prior experience and adequate training in some specific area of knowledge, such as differential and integral equation theory. However, if the expert has an idea how the dynamics works in terms of rules then she/he can simulate the dynamical behavior of the underlying phenomenon by means of a p-fuzzy system [2].

A p-fuzzy system can be viewed as a dynamic system whose the function that describes the evolution rule is given by a fuzzy rule-based systems (FRBS) [2]. P-fuzzy systems were successfully applied to estimate solutions to economic and biomathematic problems [10, 9].

In this work, we build a fuzzy Mamdani controller based on fuzzy rules that describe the behavior of the phase diagram of differential equations associated with free mechanical vibrations. Based on the proposed FRBS, we design p-fuzzy systems to model three free mechanical vibrations problems. Specifically, we focus on the dynamics of harmonic oscillators, damped vibrations, and negative damping cases.

2 Mathematical Background

2.1 Free mechanical vibrations

In a mass-spring-damper system, the differential equation is constructed using two physic laws. The first is the Hooke's law for linear springs and the Newton's second law of motion [6]. These systems are usually represented in diagrams as in Fig. 1, where $x(t)$ represent the position of a particle at the time t. Consequently, we note that $x'(t)$ and $x''(t)$ represent the velocity and acceleration of this particle in a classical mechanical vibration problem.

Here, we consider the following linear and homogeneous initial value problem (IVP) given by [8],

$$\begin{cases} mx''(t) + bx'(t) + kx(t) = 0\,, \\ x(0) = x_0\,, \\ x'(0) = y_0\,. \end{cases} \tag{1}$$

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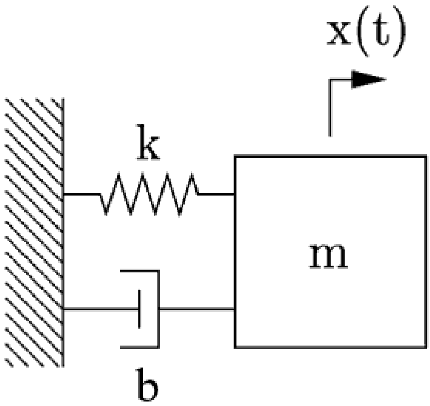

Figure 1: Free mechanical vibration diagram.

where m is the mass of a particle, b is the damping coefficient, and k is the proportionality constant (stiffness) of the spring [8].

The analytic solution of (1) is given by [8]:

$$x(t) = e^{-\beta t} \left(C_1 cos(\omega t) + C_2 sin(\omega t) \right) \qquad (2)$$

where C_1,C_2 are constant related with the initial conditions, $\beta = -\frac{b}{2m}$ is related to the amplitude decay, and $\omega = \frac{\sqrt{4mk-b^2}}{2m}$ is known as the damped natural frequency. For the undamped motion, that is, the case where $b = 0$, the natural frequency is given by $\omega_0 = \sqrt{\frac{k}{m}}$.

A special case of above differential equation occurs when the damping coefficient is negative, *i.e.*, $b < 0$, which means that the damping term imparts energy to the system [8]. In this case, we obtain unstable solutions with oscillations of growing amplitudes [8]. Under certain conditions, some solid structures could suffer the effect of negative damping. Consequently, the amplitude response may increase, leading to structural instabilities which might cause serious damage [1].

2.2 Fuzzy sets and p-fuzzy systems

A fuzzy subset of A of an universal set X is characterized by a function $\varphi_A : X \to [0,1]$ called membership function of A such that $\varphi_A(x)$ represents the membership degree of x in A [11]. For notation convenience, we also use the symbol $A(x)$ instead of $\varphi_A(x)$.

Here, we focus on a particular class of fuzzy sets, called fuzzy numbers. In particular, a trapezoidal fuzzy number A, denoted by a quadruple $(a;m;n;b)$, with $a,m,n,b \in \mathbb{R}$ and $a \leq m \leq n \leq b$, consists of a fuzzy number whose membership function is given by [2]

$$A(x) = \begin{cases} \frac{x-a}{m-a} & , \text{ if } x \in [a,m), \\ 1 & , \text{ if } x \in [m,n], \\ \frac{b-x}{b-n} & , \text{ if } x \in (n,b], \\ 0 & , \text{ otherwise.} \end{cases} \qquad (3)$$

In the case where $m = n$, we speak of triangular fuzzy number and it is denoted by the symbol $(a;m;b)$ instead of $(a;m;m;b)$ [2].

Fuzzy Rule-Based Systems (FRBS) have four components: a fuzzification module, a fuzzy rule base, a fuzzy inference method, and a defuzzification module [2, 7].

In the fuzzification module, real-valued inputs are translated into fuzzy numbers of their respective universes. Expert knowledge plays an important role to build the membership functions for each fuzzy set associated with the inputs [7].

Here, we focus on a fuzzy rule base given by a collection of fuzzy conditional rules of the form "if x_1 is A_{i1} and x_2 is A_{i2} then y is B_i", for $i = 1,\ldots,r$, where r is the number of rules, and A_{ij} and B_i, for $i = 1,\ldots,r$ and $j = 1,2$, are fuzzy sets that represent linguistic (or fuzzy) terms and are called antecedents and consequent of each fuzzy rule, respectively [2].

We use the Mamdani inference with canonical inclusion fuzzifier method. In this case, for a given input (x_1,x_2), the Mamdani inference produces as output a fuzzy set B given, $\forall y \in \mathbb{R}$, by [2]:

$$B(y) = \max_{i=1,\ldots,r} \min\{A_{i1}(x_1), A_{i2}(x_2), B_i(y)\}. \qquad (4)$$

Finally, the defuzzification module consist of a process that allows us to represent a fuzzy set by a real value. In this manuscript, we adopt the centroid scheme [7].

A *partially* fuzzy system or, for short, a p-fuzzy system, is a dynamical system generated by ordinary differential equations (ODEs) where the direction field is given by FRBS based on an *a priori* partial knowledge of the direction field. Furthermore, the state variables and their variations are considered linguistic. Thus, the state variables are correlated to their variations by means of fuzzy rules where the state variables are the input and the variations are outputs. Since in such methodologies, processes of defuzzification are expected, the final solution of a p-fuzzy system is deterministic [2]. Here, we use p-fuzzy systems to deal with autonomous initial value problems (IVPs) of the form

$$\begin{cases} x'(t) = f(x,y)\,, \\ y'(t) = g(x,y)\,, \\ x(0) = x_0\,, \\ y(0) = y_0\,. \end{cases} \qquad (5)$$

where the functions f and g are partially known. To obtain the solution of the IVP (5) via a p-fuzzy system or at least an approximation to it, without knowing the field f and g explicitly, we take advantage of the qualitative information available to design a fuzzy rule base which represent the properties that characterize the phenomenon [4]. Thus, the solution $(x(t), y(t))$ of (5) can be estimated by a sequence (X_n, Y_n) obtained by applying a numerical methods to the associated p-fuzzy system , such as the Euler or Runge Kutta methods [2]. In this work, we use the Euler method. More precisely, we use the following formulas:

$$\begin{aligned} X_{n+1} &= X_n + h\, f(X_n, Y_n)\,, \\ Y_{n+1} &= Y_n + h\, g(X_n, Y_n)\,, \end{aligned} \tag{6}$$

where h is the step (in time) and $f(X_n, Y_n)$ and $g(X_n, Y_n)$ are the variations rates estimated by the proposed FRBSs. Thus, we can rewrite (5) as follows:

$$\begin{cases} x'(t) = \mathrm{FRBS}_f(x, y)\,, \\ y'(t) = \mathrm{FRBS}_g(x, y)\,, \\ x(0) = x_0\,, \\ y(0) = y_0\,. \end{cases} \tag{7}$$

Note that, in general, a Mamdani fuzzy controller yields a function f_r^* (and g_r^*) where r denotes the number of rules in the fuzzy rule base. Thus, it seems reasonable to assume that the adjusted function f_r^* (and g_r^*) approximates f (and g) when the number of data r increases [4].

3 Solutions of Free Mechanical Vibrations Problems via P-Fuzzy Systems

Using the following changing of variables $x_1 = x(t)$ and $x_2 = x'(t)$ in IVP (1), we obtain a first order linear and autonomous system given by:

$$\begin{cases} x_1' = x_2, \\ x_2' = -\frac{k}{m}x_1 - \frac{b}{m}x_2\,, \\ x_1(0) = x_0\,, \\ x_2(0) = y_0\,. \end{cases} \tag{8}$$

Since the system (8) can be viewed as particular case of the system (5), we can obtain a numerical solution for (8) by means of an associated p-fuzzy system. In the next subsections, we present the corresponding FRBSs and some computational simulations for free mechanical vibrations models.

3.1 Fuzzy rule bases for free mechanical vibrations models

Recall that in a p-fuzzy system, the vector fields are given by FRBSs (see in Subsection 2.2). Here, we employ a p-fuzzy system of the form (7) that we use to estimate the solution of the system (8). The antecedents and consequents of the corresponding fuzzy rules are linguistic terms associated respectively with the input and output variabless. Here we use trapezoidal or triangular fuzzy numbers to represent this linguistic terms.

For the free mechanical vibration model (8), both input variables X and Y (position and velocity of a particle, respectively) can be classified as "*left*" (A_1 and B_1), "*middle left*" (A_2 and B_2), "*middle right*" (A_3 and B_3), and "*right*" (A_4 and B_4). Moreover, in the output variables, the variations rates of the input variables $\frac{dX}{dt}$ and $\frac{dY}{dt}$, can assume the fuzzy linguistic terms "*high negative*" (N_1 and M_1), "*low negative*" (N_2 and M_2), "*low positive*" (N_3 and M_3), and "*high positive*" (N_4 and M_4). Fig. 2 and Fig. 3 illustrate the form and order in which the membership functions A_i and N_i, $i = 1, \ldots, 4$, are adjusted. The membership function of B_i and M_i are adjusted in a similar form of A_i and N_i, for $i = 1, \ldots, 4$, respectively.

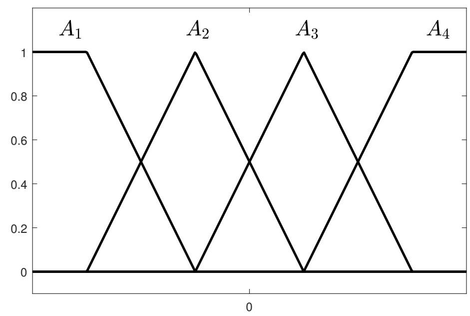

Figure 2: Antecedents of position (X) for the free mechanical vibration p-fuzzy model.

We elaborate a fuzzy rule-based system based on the differential equations for the free mechanical vibration model given as in (8). Here, we consider three cases: the harmonic oscillator ($b = 0$), the damper vibration ($b > 0$), and negative damping ($b < 0$). Roughly speaking, our strategy to obtain fuzzy rules in all cases is to analyze the variation rates of one variable considering the other variable fixed. For example, in the harmonic oscillator case ($b = 0$), the variations rates x_1' and x_2' depend on the values of x_2 and $-x_1$, respectively. Similar considerations can be established with respect to other two damping cases ($b \neq 0$). These assumptions can be translated into a set of fuzzy rules that play the role of a direction field.

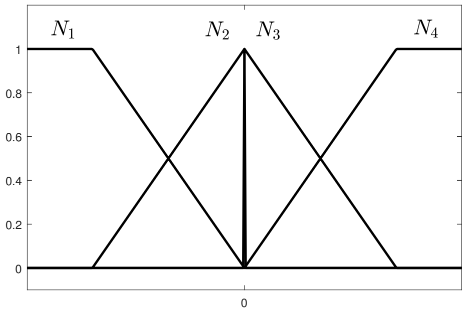

Figure 3: Consequent of position ($\frac{dX}{dt}$) for the free mechanical vibration p-fuzzy model.

Fig. 4, Fig. 5, and Fig. 6 exhibit the graphical representations of the obtained fuzzy rule base where the arrows represent the direction and magnitude of the variation rates, that is, the arrow on the right/up (left/down) side indicates positive (negative) variations and the length of the arrow indicates the magnitude of these variations.

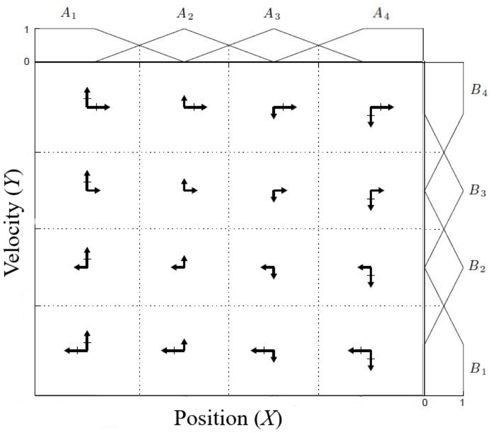

Figure 4: Graphic representation of the fuzzy rules as direction vectors for harmonic oscillator case ($b = 0$).

Fuzzy rule bases for free mechanical vibration models are established from the graphical interpretation of Fig. 4, Fig. 5, and Fig. 6 for all cases. Here, we use the symbols $\dot{X} = \frac{dX}{dt}$ and $\dot{Y} = \frac{dY}{dt}$ for the variational rates in the fuzzy rules.

Firstly, we construct a fuzzy rule base for the harmonic oscillator case ($b = 0$) consisting of 16 fuzzy rules of the type:

r_1: If X is A_1 and Y is B_1 then $\dot{X}$ is N_1 and $\dot{Y}$ is M_4.

r_2: If X is A_2 and Y is B_1 then $\dot{X}$ is N_1 and $\dot{Y}$ is M_3.

$\vdots \quad \vdots \quad \vdots$

r_8: If X is A_4 and Y is B_2 then $\dot{X}$ is N_2 and $\dot{Y}$ is M_1.

r_9: If X is A_1 and Y is B_3 then $\dot{X}$ is N_3 and $\dot{Y}$ is M_4.

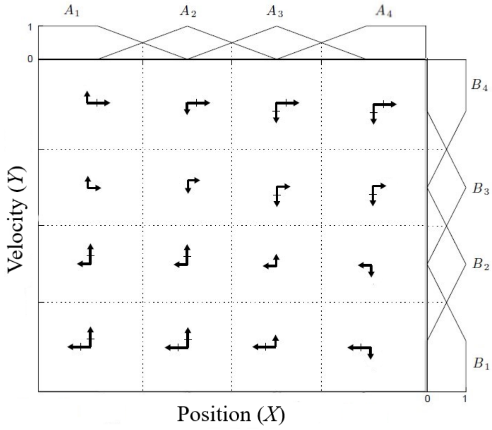

Figure 5: Graphic representation of the fuzzy rules as direction vectors for damped vibration case ($b > 0$).

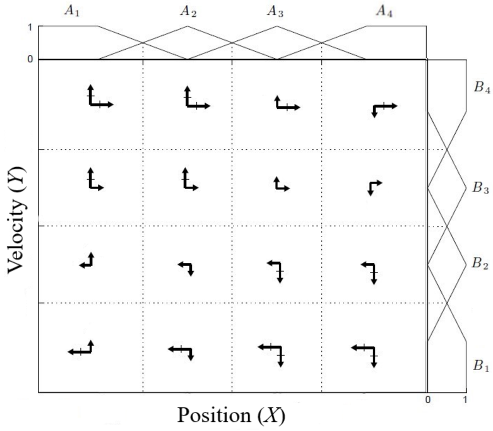

Figure 6: Graphic representation of the fuzzy rules as direction vectors for negative damping case ($b < 0$).

$\vdots \quad \vdots \quad \vdots$

r_{15}: If X is A_3 and Y is B_4 then $\dot{X}$ is N_4 and $\dot{Y}$ is M_2.

r_{16}: If X is A_4 and Y is B_4 then $\dot{X}$ is N_4 and $\dot{Y}$ is M_1.

Secondly, using Figure 5, we construct a fuzzy rule base for the damped case where $b > 0$, containing 16 fuzzy rules of the type:

r_1: If X is A_1 and Y is B_1 then $\dot{X}$ is N_1 and $\dot{Y}$ is M_4.

r_2: If X is A_2 and Y is B_1 then $\dot{X}$ is N_1 and $\dot{Y}$ is M_4.

$\vdots \quad \vdots \quad \vdots$

r_8: If X is A_4 and Y is B_2 then $\dot{X}$ is N_2 and $\dot{Y}$ is M_2.

r_9: If X is A_1 and Y is B_3 then $\dot{X}$ is N_3 and $\dot{Y}$ is M_3.

$\vdots \quad \vdots \quad \vdots$

r_{15}: If X is A_3 and Y is B_4 then $\dot{X}$ is N_4 and $\dot{Y}$ is M_1.

r_{16}: If X is A_4 and Y is B_4 then $\dot{X}$ is N_4 and $\dot{Y}$ is M_1.

Finally, we construct a fuzzy rule base with 16 fuzzy rules for the negative damping case where $b < 0$. These fuzzy rules are graphically represented in Fig. 6. Next, we present some examples of the obtained fuzzy rules:

r_1: If X is A_1 and Y is B_1 then $\dot{X}$ is N_1 and $\dot{Y}$ is M_3.

r_2: If X is A_2 and Y is B_1 then $\dot{X}$ is N_1 and $\dot{Y}$ is M_2.

$\vdots \quad \vdots \quad \vdots$

r_8: If X is A_4 and Y is B_2 then $\dot{X}$ is N_2 and $\dot{Y}$ is M_1.

r_9: If X is A_1 and Y is B_3 then $\dot{X}$ is N_3 and $\dot{Y}$ is M_4.

$\vdots \quad \vdots \quad \vdots$

r_{15}: If X is A_3 and Y is B_4 then $\dot{X}$ is N_4 and $\dot{Y}$ is M_3.

r_{16}: If X is A_4 and Y is B_4 then $\dot{X}$ is N_4 and $\dot{Y}$ is M_2.

Once the fuzzy rule bases are established, we use the Mamdani method (in the inference module) and the centroid as the defuzzification method. Fig. 7 presents the scheme of the FRBS used in this work.

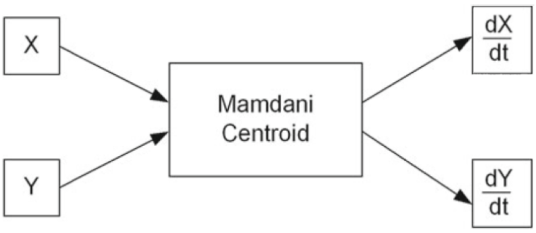

Figure 7: Fuzzy rule-based system for the free mechanical vibrations p-fuzzy models.

3.2 Results

We use the proposed p-fuzzy system to simulate the dynamic behavior of the position variable $x_1(t) = x(t)$. We compare the obtained solution with the analytic solution given by Eq. (2) for the corresponding free mechanical vibration problem.

3.2.1 Harmonic oscillator solution via p-fuzzy system

In this first case, we consider the following IVP in $t \in [0, \frac{\pi}{2}]$ given by

$$\begin{cases} \frac{1}{8}x''(t) + 16x(t) = 0\,, \\ x(0) = \frac{1}{2}\,, \\ x'(0) = \sqrt{2}\,. \end{cases} \tag{9}$$

Fig. 8 presents both p-fuzzy and analytic solutions of system (9) that represents a harmonic oscillator. For this case, with $b = 0$, we can observe that the two solutions are qualitative and quantitative similar.

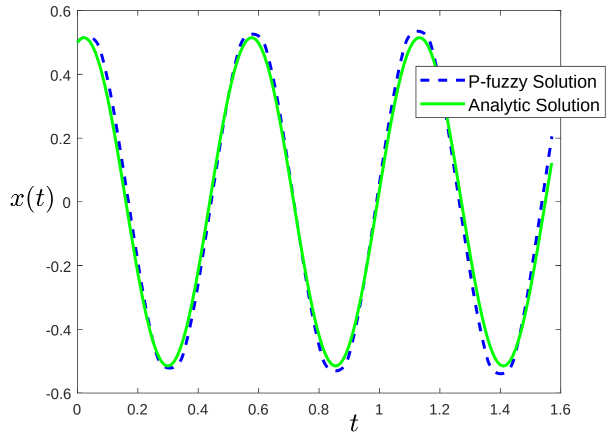

Figure 8: P-fuzzy and analytic solutions of IVP (9).

3.2.2 Damped vibrations solution via p-fuzzy system

In this second case, we consider the following IVP in $t \in [0, 25]$ given by

$$\begin{cases} 3x''(t) + x'(t) + 2x(t) = 0\,, \\ x(0) = 1\,, \\ x'(0) = 0\,. \end{cases} \tag{10}$$

Fig. 9 presents both p-fuzzy and analytic solutions of system (10) which describes a damped vibration with $b = 1$. For this case, we can observe that the both solutions are qualitative and quantitative similar.

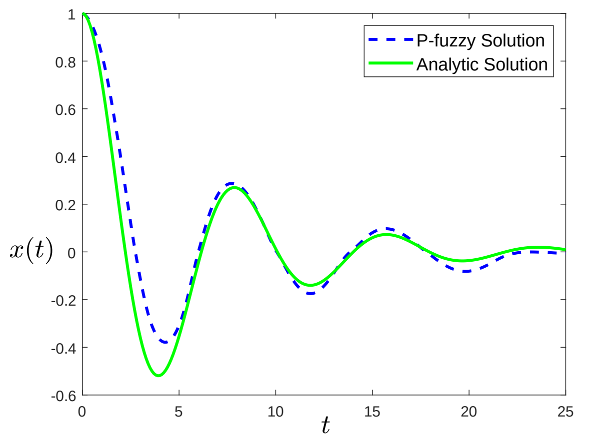

Figure 9: P-fuzzy and analytic solutions of IVP (10).

3.2.3 Negative damping solution via p-fuzzy system

In this third case, we consider the following IVP in $t \in [0, 15]$ given by

$$\begin{cases} 3x''(t) - x'(t) + 2x(t) = 0\,, \\ x(0) = 1\,, \\ x'(0) = 0\,. \end{cases} \tag{11}$$

Fig. 10 exhibits both p-fuzzy and analytic solutions of system (11). For this case, with $b = -1$, we can note that both solutions are qualitative and quantitative similar, including unstable negative damping behavior.

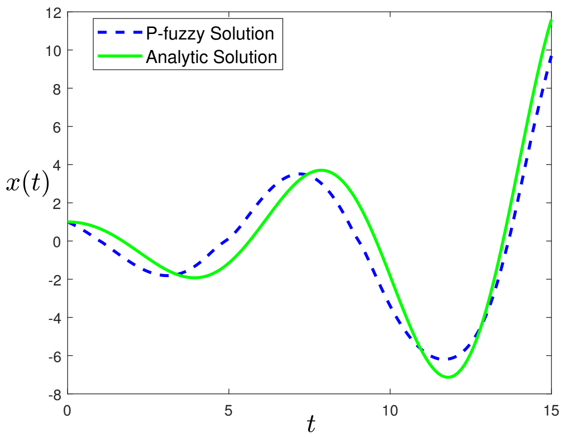

Figure 10: P-fuzzy and analytic solutions of IVP (11).

4 Concluding Remarks

The main contribution of this work is to present a p-fuzzy system that model free mechanical vibration problems.

Solutions obtained by our approach are (quantitative and qualitative) similar to the ones produced using differential equation theory. The solution yielded for the harmonic oscillator case is presented in Fig. 8. The solution for the a damped vibrations problem is presented in Fig. 9. It is worth noting that the p-fuzzy system was capable to capture the nature of the negative damping case, producing a unstable solution similar to the analytic solution as we can see in Fig. 10.

In addition, note that our proposal can be used by any person who is somehow related to physics and does not require previous experience with differential equations.

Finally, it is worth noting that the use of the p-fuzzy system is based on universal approximation capability, which means that it is a good estimator of theoretical problems.

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