New approach to the complete sum of products of the twisted (h,q)-Bernoulli numbers and polynomials

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Abstract

In this paper, by using q-Volkenborn integral[10], the first author[25] constructed new generating functions of the new twisted (h,q)-Bernoulli polynomials and numbers. We define higher-order twisted (h,q)-Bernoulli polynomials and numbers. Using these numbers and polynomials, we obtain new approach to the complete sums of products of twisted (h,q)-Bernoulli polynomials and numbers. p-adic q-Volkenborn integral is used to evaluate summations of the following form:

$$B_{m,w}^{(h,v)}(y_1 + y_2 + ... + y_v, q)$$
$$= \sum_{\substack{l_1, l_2, ..., l_v \geq 0 \\ l_1 + l_2 + ... + l_v = m}} \binom{m}{l_1, l_2, ..., l_v} \prod_{j=1}^{v} B_{l_j,w}^{(h)}(y_j, q),$$

where $B_{m,w}^{(h)}(y_j, q)$ is the twisted (h,q)-Bernoulli polynomials. We also define new identities involving (h,q)-Bernoulli polnomials and numbers.

1   Introduction, Definitions and Notations

It is purpose of this paper to define higher-order *twisted* (h,q)-Bernoulli numbers and polynomials and present the fundamental properties of them in p-adic analysis. In [10], [11], Kim discussed several ways to set up a p-adic q-intergration theory for functions on $\mathbb{Z}_p$ (for p-adic intergration and its applications see also [19], [28]). By using the p-adic q-intergral (or q-Volkenborn integral), which is given in (1.1) and (1.2), we define generating functions of the higher-order twisted (h,q)-Bernoulli numbers and polynomials. The

complete sums of products of (Carlitz's q-)Bernoulli polynomials and numbers mentioned in [5] and [8]. The motivation of this paper is to give some relations and formulae of the higher-order twisted (h, q)-Bernoulli polynomials and numbers and obtain complete sums of products of these polynomials and numbers which generalize those of [5] and [8] as well. This numbers and polynomials are used not only Number Theory, Complex Analysis, and the other branch of mathematics but also in other parts of the p-adic Analysis and Mathematical Physics ([1], [2], [10], [11], [18], [19], [27], [28]).

Twisted version of Bernoulli numbers and polynomials were studied by many authors. We introduce some of them here. Koblitz[18] defined usual twisted Bernoulli numbers and polynomials. He gave relations between these numbers and twisted L-functions at non-positive integers (see [17], [20], [18], [22]). Kim[6] treated analogue of Bernoulli numbers, which are called twisted Bernoulli numbers in this paper. Kim[7] constructed p-adic q-integral. He proved that Carlitz 's q-Bernoulli numbers can be represented as a p-adic q-integral by the q-analogue of the ordinary p-adic invariant measure. In [21], [23], the first author constructed generating functions of q-generalized Euler numbers and polynomials and twisted q-generalized Euler numbers and polynomials. He also constructed a complex analytic twisted l-series which is interpolated twisted q-Euler numbers at non-positive integers. Kim, Jang, Rim and Pak[16] defined twisted q- Bernoulli numbers by using p-adic invariant integrals on $\mathbb{Z}_p$. They constructed twisted q-zeta function and q-L-series which interpolate twisted q-Bernoulli numbers. In [20], [22], the first author studied twisted Bernoulli polynomials, numbers and analytic properties of twisted L-functions. He gave the relation between twisted Bernoulli numbers and twisted L-functions at non-positive integers. He also defined q-analogues of the twisted Bernoulli polynomials and twisted L-functions. In [24], the first author defined generating functions of q-Bernoulli numbers and polynomials. By applying the Mellin transformation to these functions, he constructed q-zeta function, q-L-functions and q-Dedekind type sums. In [10], [11] and [14], by using q-Volkenborn integration, Kim constructed the new (h, q)-extension of the Bernoulli numbers and polynomials. He defined (h, q)-extension of the zeta functions which are interpolated new (h, q)-extension of the Bernoulli numbers and polynomials. In [25], the first author define twisted (h, q)-Bernoulli numbers, zeta functions and L-function. He also gave relations between these functions and numbers.

Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote by the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $\mid p \mid_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we normally assume

$$\mid q - 1 \mid_p < p^{-\frac{1}{p-1}},$$

so that

$$q^x = \exp(x \log q) \text{ for } \mid x \mid_p \leq 1.$$

If $q \in \mathbb{C}$, then we normally assume $\mid q \mid < 1$.

Kurt Hensel (1861-1941) invented the so-called p-adic numbers around the end of the nineteenth century. In spite of their being already one hundred years old, these numbers are

still today enveloped in an aura of mystery within scientific community [19]. Although they have penetrated several mathematical fields, Number Theory, Algebraic Geometry, Algebraic Topology, Analysis, Mathematical Physics, String Theory, Field Theory, Stochastic Differential Equations on real Banach Spaces and Manifolds and other parts of the natural sciences which are turbulence theory, dynamical systems, statistical physics, biology, etc (see for detail [1], [11], [28]). While solving mathematical and physical problems and while constructing and investigating measures on manifolds, the p-adic numbers are used. There is an unexpected connection of the p-adic Analysis with q-Analysis, Quantum Groups and Noncommutative Geometry (see [9], [10], [11], [12], [13], [27], [28]). The p-adic q-intergral (or q-Volkenborn integral) are originately constructed by Kim[9], [10], [11]. Kim[11] indicated a connection between the q-Volkenborn integral, multiple Changhee q-Bernoulli polynomials and non-Archimedean combinatorial analysis. The q-Volkenborn integral is used in mathematical physics for example the functional equation of the q-zeta function, the q-Stirling numbers, and q-Mahler theory of integration with respect to the ring $\mathbb{Z}_p$ together with Iwasawa's p-adic q-L-function. Recently, the first author [25], [26] has studied on applications of the q-Volkenborn integral.

For

$$f \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{f \mid f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the p-adic q-integral (or q-Volkenborn integration) was defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} q^x f(x),$$

where

$$[x]_q = \frac{1-q^x}{1-q}.$$

$$I_1(f) = \lim_{q \to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x). \tag{1.1}$$

If we take $f_1(x) = f(x+1)$ in (1.1), then we have

$$I_1(f_1) = I_1(f) + f'(0), \tag{1.2}$$

where $f'(0) = \frac{d}{dx}f(x)\mid_{x=0}$, cf. ([3], [9], [10], [11], [12], [13], [15], [27]).

Let p be a fixed prime. For a fixed positive integer f with $(p, f) = 1$, we set

$$
\begin{aligned}
X &= X_f = \varprojlim_N \mathbb{Z}/fp^N\mathbb{Z}, \\
X_1 &= \mathbb{Z}_p, \\
X^* &= \bigcup_{\substack{0 < a < fp \\ (a,p)=1}} a + fp\mathbb{Z}_p
\end{aligned}
$$

and

$$a + fp^N \mathbb{Z}_p = \left\{ x \in X \mid x \equiv a (\mathrm{mod} fp^N) \right\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < fp^N$. For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$,

$$\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \int_{\mathbb{X}} f(x) d\mu_1(x) \text{ cf. ([9],[11],[12]).} \tag{1.3}$$

Let

$$T_p = \underset{n \geq 1}{\cup} C_{p^n} = \lim_{\overrightarrow{n}} C_{p^n},$$

where $C_{p^n} = \left\{ w \mid w^{p^n} = 1 \right\}$ is the cyclic group of order p^n. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \to w^x$ (see [6], [16]).

By using q-Volkenborn integration (see for detail [9], [10], [11], [12], [14], [27]), the first author defined generating function of the new twisted (h, q)-extension of Bernoulli numbers, $B_{n,w}^{(h)}(q)$ by means of the following generating function[25]:

$$
\begin{aligned}
F_{w,q}^{(h)}(t) &= I_1(w^x q^{hx} e^{tx}) \\
&= \frac{\log q^h + t}{wq^h e^t - 1} \\
&= \sum_{n=0}^{\infty} B_{n,w}^{(h)}(q) \frac{t^n}{n!}
\end{aligned}
\tag{1.4}
$$

for $\mid t \mid < p^{-\frac{1}{p-1}}$ and h is an integer.

By using the above equation, and following the usual convention of symbolically replacing $(B_w^{(h)}(q))^n$ by $B_{n,w}^{(h)}(q)$, we have

$$
\begin{aligned}
B_{0,w}^{(h)}(q) &= \frac{\log q^h}{wq^h - 1} \\
wq^h (B_w^{(h)}(q) + 1)^n - B_{n,w}^{(h)}(q) &= \delta_{1,n}, \ n \geq 1,
\end{aligned}
\tag{1.5}
$$

where $\delta_{1,n}$ is denoted Kronecker symbol (see [26]).

We note that if $w \to 1$, then $B_{n,w}^{(h)}(q) \to B_n^{(h)}(q)$ and $F_{w,q}^{(h)}(t) \to F_q^{(h)}(t) = \frac{h \log q + t}{q^h e^t - 1}$. If $q \to 1, h = 1$, then $F_q^{(h)}(t) \to F(t) = \frac{t}{e^t - 1} = \sum_{n=1}^{\infty} B_n \frac{t^n}{n!}$, where B_n is usual Bernoulli numbers, cf. ([14], [22], [27]).

Twisted version of Witt's formula for $B_{n,w}^{(h)}(q)$ is given by the following theorem [25]:

Theorem 1. *For $h \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $\mid q - 1 \mid_p < p^{-\frac{1}{p-1}}$, we have*

$$B_{n,w}^{(h)}(q) = \int_{\mathbb{Z}_p} \phi_w(x) q^{hx} x^n d\mu_1(x). \tag{1.6}$$

Twisted (h, q)-extension of Bernoulli polynomials $B_{n,w}^{(h)}(z, q)$ are defined by means of the generating function

$$
\begin{aligned}
F_{w,q}^{(h)}(t, z) &= \frac{(t + \log q^h)e^{tz}}{wq^h e^t - 1} \\
&= I_1(\phi_w(x)q^{hx}e^{t(z+x)}) \\
&= \sum_{n=0}^{\infty} B_{n,w}^{(h)}(z, q)\frac{t^n}{n!} \quad \text{cf. [25].}
\end{aligned}
\tag{1.7}
$$

We note that $B_{n,w}^{(h)}(0, q) = B_{n,w}^{(h)}(q)$. If $w \to 1$, then $B_{n,w}^{(h)}(z, q) \to B_n^{(h)}(z, q)$ and $F_{w,q}^{(h)}(t, z) \to F_q^{(h)}(t, z) = F_q^{(h)}(t)e^{tz}$ (see [14]). By applying Mellin transformation to (1.7), the first author[25] constructed twisted (h, q)-Hurwitz zeta functions and twisted (h, q)-L-functions.

Twisted version of Witt's formula for $B_{n,w}^{(h)}(z, q)$ is given by the following theorem [25]:

Theorem 2. *For $h \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $\mid q - 1 \mid_p < p^{-\frac{1}{p-1}}$, we obtain*

$$
B_{n,w}^{(h)}(z, q) = \int_{\mathbb{Z}_p} \phi_w(x)q^{hx}(x + z)^n d\mu_1(x).
\tag{1.8}
$$

Observe that if $w \to 1, q \to 1, h = 1$, then $B_{n,w}^{(h)}(z, q) \to B_n(z)$, the usual Bernoulli polynomials, cf. ([14], [22], [27]). If $w \to 1$, then (1.8) reduces to (1.9).

Theorem 3. *([25]) For $n \geq 0$, we have*

$$
B_{n,w}^{(h)}(z, q) = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} B_{k,w}^{(h)}(q).
$$

In section 2 we shall define higher-order of the (h, q)-extension of twisted Bernoulli numbers and polynomials. We shall also derive some properties of them in detail.

Observe that letting $w \to 1$ in Theorem 10 and (1.7), we have

$$
\frac{(t + \log q^h)e^{tz}}{q^h e^t - 1} = \sum_{n=0}^{\infty} B_n^{(h)}(z, q)\frac{t^n}{n!} \quad \text{cf ([14]).}
$$

Put $z = 0$, in the above

$$
\frac{t + \log q^h}{q^h e^t - 1} = \sum_{n=0}^{\infty} B_n^{(h)}(q)\frac{t^n}{n!}.
$$

We note that $B_n^{(h)}(0, q) = B_n^{(h)}(q)$. By using Theorem 10, we easily see that

$$
B_n^{(h)}(z, q) = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} B_k^{(h)}(q) \quad \text{cf ([14]).}
$$

For detail on the twisted (h, q)-Bernoulli polynomials and numbers, $B_{n,w}^{(h)}(z, q)$, $B_{n,w}^{(h)}(q)$, (or (h, q)-Bernoulli polynomials and numbers ,$B_n^{(h)}(z, q)$, $B_n^{(h)}(q)$), see (1.5) and cf. ([14], [25]).

Witt formula of $B_n^{(h)}(q)$ and $B_n^{(h)}(z, q)$ is given by the following theorem:

Theorem 4. *([14]) For $h \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $\mid 1 - q \mid_p < p^{-\frac{1}{p-1}}$, we have*

$$\int_{\mathbb{Z}_p} q^{hx} x^n d\mu_1(x) = B_{n,}^{(h)}(q),$$

$$\int_{\mathbb{Z}_p} q^{hy}(x + y)^n d\mu_1(y) = B_n^{(h)}(x, q). \tag{1.9}$$

Observe that if $w \to 1$, then Theorem 1 and Theorem 2 reduce to Theorem 4.

This paper is organized as follows:

In this paper, by making multiple use of p-adic q-integral, we construct generating functions of the higher-order twisted (h, q)-Bernoulli polynomials and numbers. We generalize Theorem 2 and Theorem 3. We prove complete sums of products of higher-order twisted (h, q)-Bernoulli polynomials and numbers. We also find new relations involving higher-order twisted (h, q)-Bernoulli polynomials and numbers.

2 Higher-order twisted (h, q)-Bernoulli polynomials and numbers

Our primary aim in this section is to give complete sums of products of twisted (h, q)-Bernoulli polynomials and numbers. Therefore, we firstly need generating functions of this numbers and polynomials. By using multiple p-adic q-integral, higher-order twisted (h, q)-Bernoulli numbers, $B_{n,w}^{(h,v)}(q)$ are defined by means of the following generating function:

$$\underbrace{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} ... \int_{\mathbb{Z}_p}}_{v-times} (q^h w)^{\sum_{j=1}^v x_j} \exp\left(t \sum_{j=1}^v x_j\right) \prod_{j=1}^v d\mu_1(x_j)$$

$$= \sum_{n=0}^{\infty} B_{n,w}^{(h,v)}(q) \frac{t^n}{n!}. \tag{2.1}$$

Observe that when $v = 1$, $\lim_{q \to 1} B_{n,w}^{(h,v)}(q) = B_{n,w}^{(h)}$ (see [25]). If $h = 1$, $\lim_{w \to 1} B_n^{(v)}(q) = B_n^{(v)}(q)$ (see [27]).

By using Taylor series of $\exp(tx)$ in (2.1), we have

$$\sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} ... \int_{\mathbb{Z}_p} (q^h w)^{\sum_{j=1}^v x_j} \left(\sum_{j=1}^v x_j\right)^n \prod_{j=1}^v d\mu_1(x_j)\right) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} B_{n,w}^{(h,v)}(q) \frac{t^n}{n!}.$$

By comparing coefficients $\frac{t^n}{n!}$ in both sides of the above equation, after simple calculation, we arrive at the following theorem, which is known Witt-type formula of $B_{n,w}^{(h,v)}(q)$:

Theorem 5. *For $h \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $\mid q - 1 \mid_p < p^{-\frac{1}{p-1}}$, we have*

$$B_{n,w}^{(h,v)}(q) = \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} ... \int_{\mathbb{Z}_p} (q^h w)^{\sum_{j=1}^{v} x_j} \left(\sum_{j=1}^{v} x_j \right)^n \prod_{j=1}^{v} d\mu_1(x_j). \tag{2.2}$$

Observe that substituting $v = 1$ into (2.2), Theorem 5, reduces to Theorem 2. Letting $w \to 1$ in (2.1), we see that

$$\underbrace{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} ... \int_{\mathbb{Z}_p}}_{v-times} q^{h \sum_{j=1}^{v} x_j} \exp \left(t \sum_{j=1}^{v} x_j \right) \prod_{j=1}^{v} d\mu_1(x_j)$$

$$= \sum_{n=0}^{\infty} B_n^{(h,v)}(q) \frac{t^n}{n!},$$

where $B_n^{(h,v)}(q)$ denotes the higher-order q-Bernoulli numbers. By comparing coefficients $\frac{t^n}{n!}$ in both sides of the above equation, we have

$$B_n^{(h,v)}(q) = \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} ... \int_{\mathbb{Z}_p} q^{h \sum_{j=1}^{v} x_j} \left(\sum_{j=1}^{v} x_j \right)^n \prod_{j=1}^{v} d\mu_1(x_j).$$

The above relation was proved by Kim[8] related to the higher-order Carlitz's q-Bernoulli numbers.

By using (2.1), higher-order twisted (h, q)-Bernoulli numbers, $B_{n,w}^{(h,v)}(q)$ are defined by the following generating function:

$$\left(\frac{\log q^h + t}{wq^h e^t - 1} \right)^v = \sum_{n=0}^{\infty} B_{n,w}^{(h,v)}(q) \frac{t^n}{n!}. \tag{2.3}$$

Observe that substituting $v = 1$ into (2.3), then we arrive at (1.4).

Higher-order twisted (h, q)-Bernoulli polynomials, $B_{n,w}^{(h,v)}(z, q)$ are defined by means of the following generating function:

$$\underbrace{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} ... \int_{\mathbb{Z}_p}}_{v-times} (q^h w)^{\sum_{j=1}^{v} x_j} \exp \left(tz + \sum_{j=1}^{v} tx_j \right) \prod_{j=1}^{v} d\mu_1(x_j)$$

$$= \sum_{n=0}^{\infty} B_{n,w}^{(h,v)}(z, q) \frac{t^n}{n!}.$$

By using Taylor series of $\exp(tx)$ in the above equation, we have

$$\sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} ... \int_{\mathbb{Z}_p} (q^h w)^{\sum_{j=1}^{v} x_j} \left(z + \sum_{j=1}^{v} x_j \right)^n \prod_{j=1}^{v} d\mu_1(x_j) \right) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} B_{n,w}^{(h,v)}(z, q) \frac{t^n}{n!}.$$

By comparing coefficients $\frac{t^n}{n!}$ in both sides of the above equation, after simple calculation, we arrive at the following theorem, which is known Witt-type formula of $B_{n,w}^{(h,v)}(z,q)$:

Theorem 6. *For $h \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $\mid q - 1 \mid_p < p^{-\frac{1}{p-1}}$, we have*

$$B_{n,w}^{(h,v)}(z,q) = \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (q^h w)^{\sum_{j=1}^{v} x_j} \left(z + \sum_{j=1}^{v} x_j \right)^n \prod_{j=1}^{v} d\mu_1(x_j). \tag{2.4}$$

Letting $w \to 1$ in (2.1), we obtain

$$\underbrace{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{v-times} q^{h\sum_{j=1}^{v} x_j} \exp t \left(z + \sum_{j=1}^{v} x_j \right) \prod_{j=1}^{v} d\mu_1(x_j)$$

$$= \sum_{n=0}^{\infty} B_n^{(h,v)}(z,q) \frac{t^n}{n!},$$

where $B_n^{(h,v)}(z,q)$ denotes the higher-order q-Bernoulli numbers. By comparing coefficients $\frac{t^n}{n!}$ in both sides of the above equation, we have

$$B_n^{(h,v)}(z,q) = \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{h\sum_{j=1}^{v} x_j} \left(z + \sum_{j=1}^{v} x_j \right)^n \prod_{j=1}^{v} d\mu_1(x_j).$$

By (2.1), higher-order twisted (h,q)-Bernoulli polynomials, $B_{n,w}^{(h,v)}(q)$ are defined by the following generating function

$$F_{q,w}^{h,v}(z,t) = \exp(tz) \left(\frac{\log q^h + t}{wq^h e^t - 1} \right)^v = \sum_{n=0}^{\infty} B_{n,w}^{(h,v)}(z,q) \frac{t^n}{n!}. \tag{2.5}$$

Note that letting $w \to 1$, $h = 1$, $q \to 1$ in (2.5), we see that

$$F^{1,v}(z,t) = \exp(tz) \left(\frac{t}{e^t - 1} \right)^v = \sum_{n=0}^{\infty} B_n^{(v)}(z) \frac{t^n}{n!},$$

where $B_n^{(v)}(z)$ denote classical higher-order Bernoulli polynomials [27].

The higher-order twisted (h,q)-Bernoulli polynomials, $B_{n,w}^{(h,v)}(q)$ are given by the following theorem, explicitly:

Theorem 7. *Let n be a positive integer. Then we have*

$$B_{n,w}^{(h,v)}(z,q) = \sum_{l=0}^{n} \binom{n}{l} z^{n-l} B_{l,w}^{(h,v)}(q). \tag{2.6}$$

Proof. By using binomial expansion in (2.4), we have

$$B_{n,w}^{(h,v)}(z,q) = \sum_{l=0}^{n} \binom{n}{l} z^{n-l} \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (q^h w)^{\sum_{j=1}^{v} x_j} \left(\sum_{j=1}^{v} x_j \right)^l \prod_{j=1}^{v} d\mu_1(x_j).$$

By (2.2) in the above, we arrive at the desired result. ∎

We note that $B_{n,w}^{(h,v)}(1,q) = B_{n,w}^{(h,v)}(0,q) = B_{n,w}^{(h,v)}(q)$.

We need the Multinomial Theorem, which is given as follows cf. ([2], [4]):

Theorem 8. *(Multinomial Theorem)*

$$\left(\sum_{j=1}^{v} x_j\right)^n = \sum_{\substack{l_1,l_2,...,l_v \geq 0 \\ l_1 + l_2 + ... + l_v = n}} \left(\begin{array}{c} n \\ l_1, l_2, ..., l_v \end{array}\right) \prod_{a=1}^{v} x_a^{l_a},$$

where $\left(\begin{array}{c} n \\ l_1, l_2, ..., l_v \end{array}\right)$ *are the multinomial coefficients, which are defined by*

$$\left(\begin{array}{c} n \\ l_1, l_2, ..., l_v \end{array}\right) = \frac{n!}{l_1! l_2! ... l_v!}.$$

Theorem 9.

$$B_{n,w}^{(h,v)}(q) = \sum_{\substack{l_1,l_2,...,l_v \geq 0 \\ l_1 + l_2 + ... + l_v = n}} \left(\begin{array}{c} n \\ l_1, l_2, ..., l_v \end{array}\right) \prod_{j=1}^{v} B_{l_j,w}^{(h)}(q). \tag{2.7}$$

Proof. By using Theorem 8 in (2.2), we have

$$B_{n,w}^{(h,v)}(q)$$

$$= \sum_{\substack{l_1,l_2,...,l_v \geq 0 \\ l_1 + l_2 + ... + l_v = n}} \left(\begin{array}{c} n \\ l_1, l_2, ..., l_v \end{array}\right) \prod_{j=1}^{v} \int_{\mathbb{Z}_p} \left(q^h w\right)^{x_j} x_j^{l_j} d\mu_1(x_j).$$

By (1.6) in the above, we obtain the desired result.   ∎

Remark 1.

$$\lim_{\substack{q \to 1 \\ h = 1}} B_{n,w}^{(h,v)}(q) = B_{n,w}^{(v)} = \sum_{\substack{l_1,l_2,...,l_v \geq 0 \\ l_1 + l_2 + ... + l_v = n}} \left(\begin{array}{c} n \\ l_1, l_2, ..., l_v \end{array}\right) B_{l_1,w} B_{l_2,w} ... B_{l_v,w},$$

and

$$\lim_{w \to 1} B_{n,w}^{(v)} = B_n^{(v)} = \sum_{\substack{l_1,l_2,...,l_v \geq 0 \\ l_1 + l_2 + ... + l_v = n}} \left(\begin{array}{c} n \\ l_1, l_2, ..., l_v \end{array}\right) B_{l_1} B_{l_2} ... B_{l_v},$$

where $B_{l_1,w}$ and B_{l_1} denote classical twisted Bernoulli numbers and classical Bernoulli numbers, respectively.

By substituting (2.7) into (2.6), after some elementary calculations, we arrive at the following corollary:

Corollary 1.

$$B_{n,w}^{(h,v)}(z,q) = \sum_{m=0}^{n} \sum_{\substack{l_1, l_2, ..., l_v \geq 0 \\ l_1 + l_2 + ... + l_v = m}} \binom{n}{m} \binom{m}{l_1, l_2, ..., l_v} z^{n-m} \prod_{j=1}^{v} B_{l_j,w}^{(h)}(q).$$

One of the main theorem of this section is to give complete sum of products of the higher-order twisted (h,q)-Bernoulli polynomials. Our results are given in the following theorem:

Theorem 10. *For $y_1, y_2,..., y_v \in \mathbb{C}_p$ and positive integers n, v, we have*

$$B_{m,w}^{(h,v)}(y_1 + y_2 + ... + y_v, q) = \sum_{\substack{l_1, l_2, ..., l_v \geq 0 \\ l_1 + l_2 + ... + l_v = m}} \binom{m}{l_1, l_2, ..., l_v} \prod_{j=1}^{v} B_{l_j,w}^{(h)}(y_j, q).$$

Proof. By substituting $z = y_1 + y_2 + ... + y_v$ into (2.4), we have

$$B_{n,w}^{(h,v)}(y_1 + y_2 + ... + y_v, q) = \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} ... \int_{\mathbb{Z}_p} (q^h w)^{\sum_{j=1}^{v} x_j} \left(\sum_{j=1}^{v} (y_j + x_j) \right)^n \prod_{j=1}^{v} d\mu_1(x_j).$$

By using Theorem 8 in the above, and after some elementary calculations, we get

$$B_{m,w}^{(h,v)}(y_1 + y_2 + ... + y_v, q)$$

$$= \sum_{\substack{l_1, l_2, ..., l_v \geq 0 \\ l_1 + l_2 + ... + l_v = m}} \binom{m}{l_1, l_2, ..., l_v} \prod_{j=1}^{v} \int_{\mathbb{Z}_p} (q^h w)^{x_j} (y_j + x_j)^{l_j} d\mu_1(x_j).$$

By substituting (1.8) into the above, we arrive at the desired result. ∎

Observe that letting $w \to 1$ in Theorem 10, we have

$$B_n^{(h,v)}(y_1 + y_2 + ... + y_v, q) = \sum_{\substack{l_1, l_2, ..., l_v \geq 0 \\ l_1 + l_2 + ... + l_v = m}} \binom{m}{l_1, l_2, ..., l_v} \prod_{j=1}^{v} B_{l_j}^{(h)}(y_j, q). \quad (2.8)$$

Remark 2. If we take $y_1 = y_2 = ... = y_v = 0$ in Theorem 10, then Theorem 10 reduces to Theorem 9. Substituting $h = 1$ and $q \to 1$ into (2.8), we obtain the following relation which was proved by I.-C. Huang and S.-Y. Huang [5]:

$$B_n^{(v)}(y_1 + y_2 + ... + y_v, q) = \sum_{\substack{l_1, l_2, ..., l_v \geq 0 \\ l_1 + l_2 + ... + l_v = m}} \binom{m}{l_1, l_2, ..., l_v} \prod_{j=1}^{v} B_{l_j}(y_j).$$

Kim[8] defined Carlitz's q-Bernoulli number of higher order using an integral by the q-analogue μ_q[7] of the ordinary p-adic invariant measure. By using q-Bernoulli numbers of higher order, he gave another proof of the above formula.

Using the following multinomial relations cf. ([2] pp. 25, 56), ([4] pp. 168)

$$(x + y + z)^n = \sum_{\substack{0 \le l_1, l_2, l_3 \le n \\ l_1 + l_2 + l_3 = n}} \frac{(l_1 + l_2 + l_3)!}{l_1! l_2! l_3!} x^{l_1} y^{l_2} z^{l_3}$$

$$= \sum_{\substack{0 \le l_1, l_2, l_3 \le n \\ l_1 + l_2 + l_3 = n}} \binom{l_1 + l_2 + l_3}{l_2 + l_3} \binom{l_2 + l_3}{l_3} x^{l_1} y^{l_2} z^{l_3},$$

and

$$\binom{l_1 + l_2 + \ldots + l_v}{l_1, l_2, \ldots, l_v} = \frac{(l_1 + l_2 + \ldots + l_v)!}{l_1! l_2! \ldots l_v!}$$

$$= \binom{l_1 + l_2 + l_3 + \ldots + l_v}{l_2 + l_3 + \ldots + l_v} \cdots \binom{l_{v-1} + l_v}{l_v}.$$

and our methods, one can also evaluate the other sums (see [5], [8]).

Theorem 11. *Let $n \in \mathbb{N}$. Then we have*

$$B_{n,w}^{(h,v)}(z + y, q) = \sum_{l=0}^{n} \binom{n}{l} B_{l,w}^{(h,v)}(y, q) z^{n-l}.$$

Proof.

$$B_{n,w}^{(h,v)}(z + y, q) = \left(B_w^{(h,v)} + z + y \right)^n$$

$$= \sum_{l=0}^{n} \binom{n}{l} \left(B_w^{(h,v)}(q) \right)^l (y + z)^{n-l}$$

$$= \sum_{l=0}^{n} \binom{n}{l} B_{l,w}^{(h,v)}(q) \sum_{m=0}^{n-l} \binom{n-l}{m} y^m z^{n-l-m}$$

$$= \sum_{l=0}^{n} \binom{n}{l} B_{l,w}^{(h,v)}(q) \sum_{m=l}^{n-l} \binom{n-l}{m-l} y^{m-l} z^{n-m}$$

$$= \sum_{0 \le l \le m \le n} \binom{n}{l} \binom{n-l}{m-l} B_{l,w}^{(h,v)}(q) y^{m-l} z^{n-m}$$

$$= \sum_{0 \le l \le m \le n} \binom{n}{m} \binom{m}{l} B_{l,w}^{(h,v)}(q) y^{m-l} z^{n-m}$$

$$= \sum_{m=0}^{n} \binom{n}{m} \left(\sum_{l=0}^{m} \binom{m}{l} B_{l,w}^{(h,v)}(q) y^{m-l} \right) z^{n-m},$$

with usual convention of symbolically replacing $B_w^{n(h,v)}$ by $B_{n,w}^{(h,v)}$. By using (2.6) in the above, we have

$$B_{n,w}^{(h,v)}(z + y, q) = \sum_{m=0}^{n} \binom{n}{m} B_{m,w}^{(h,v)}(y, q) z^{n-m}.$$

Thus the proof is completed. ∎

By using Theorem 10 and Theorem 11, we easily arrive at the following interesting result:

Corollary 2. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{m=0}^{n} \binom{n}{m} B_{m,w}^{(h,v)}(y_1, q) y_2^{n-m} = \sum_{\substack{l_1, l_2 \geq 0 \\ l_1 + l_2 = n}} \binom{n}{l_1, l_2} B_{l_1,w}^{(h)}(y_1, q) B_{l_2,w}^{(h)}(y_2, q).$$

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