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Generalized Distance and an Example of Fuzzy Metric

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Abstract

In this paper we introduce a new notion of generalized metric, called i-metric. This generalization is made by changing the valuation space of the distance function. The result is an interesting distance function for the set of fuzzy numbers of Interval Type with non negative fuzzy numbers as values. This example of i-metric generates a topology in a very natural way, based on open balls. We prove that this topology is Hausdorff, regular but not metrizable(generated by an usual metric).

Keywords: i-Metric, Fuzzy Metric, Topology, Metrizability.

1. Introduction

There are several notions of fuzzy metrics, as seen in [1], [2], [3]. In the references [2] and [4] the authors proved that the resulting topology is metrizable. In this paper, we introduce a notion of generalized metric and provide an example on fuzzy numbers (interval type) whose the generated topology is Hausdorff, regular, but not metrizable. This example is interesting in the field of topology, since the construction of this topology is very similar to the construction of the usual metric case and the generalized metric has relation with the important concept of interval representation, as explained below.

This paper is structured in the following way: Section 1 presents some required order theory concepts and the notion of i-metric and i-metric valuation; Section 3 presents the topology of an i-metric space; Section 4 presents a brief introduction on fuzzy numbers; Section 5 presents the example of fuzzy metric; Section 6 presents the properties of the topology resulting from the i-metric presented in section 5 and section 7 presents the final remarks.

2. i-Metric Valuation and i-Metric

In this first section, we construct the codomain of the new distance. The basic prerequisites for this section are order theory and domain theory(see [6] and [7]). Some of the notion presented here are new: semi-auxiliary relation, separable smallest element and IDV (short for i-distance valuation).

Definition 2.1 *Let $\leq$ be a partial order on A (in this case $\langle A, \leq\rangle$ is called a poset). A binary relation R on A is a* ***semi-auxiliary relation to*** *$\leq$ when:*

1. *$aRb \Rightarrow a \leq b$;*
2. *If $a \leq b$, bRc and $c \leq d$, then aRd.*

This definition is very similar to the definition of auxiliary relation (see [7]). The reason to provide this weaker concept is that the strict relation $<$ is not an auxiliary relation to $\leq$ on posets with smallest element $\perp$, since $\perp \not< \perp$.

Proposition 2.1 *If $\langle A, \leq\rangle$ is a poset, then the strict relation $a < b \Leftrightarrow (a \leq b) \wedge (a \neq b)$ is a semi-auxiliary relation to $\leq$.*

Proof: The first condition is immediate. To prove the second, suppose that $a \leq b$, $b < c$ and $c \leq d$. Since $b < c$, we have $b \leq c$ and, by the transitivity of $\leq$, we have $a \leq d$. It only remains to verify that $a \neq d$. Suppose that $a = d$. Thus $b = c$, which contradicts the hypothesis $b < c$. Thus, we must have $a \neq d$ so $a < d$.

■

Proposition 2.2 *If $\langle A, \leq\rangle$ is a poset, then every semi-auxiliary relation to $\leq$ is transitive.*

Proof: Let R be a semi-auxiliary relation to $\leq$ and suppose that aRb and bRc. Thus, we have $a \leq b$, bRc and $c \leq c$, so, from the second condition, it follows that aRc.

■

Definition 2.2 *A poset with smallest element and a semi-auxiliary relation R, $\langle A, \leq, R, \perp\rangle$ is said to have* ***separable smallest element****, when A is d-directed(every subset of A with two elements has lower bound) and for every pair of elements $a, b \in A$, with $\perp Ra$ and $\perp Rb$, there is a lower bound c for $\{a, b\}$ such that $\perp Rc$.*

Example 2.1 *Not every poset in the conditions of above definition $\langle A, \leq, \perp_A\rangle$ has separable smallest element. Consider $\mathbb{N}^* = \{1, 2, ...\}$, the partial order $a \leq_d b \Leftrightarrow a|b$ and its strict relation $<$. The smallest element of $\langle \mathbb{N}^*, \leq_d\rangle$ is 1 and for $a, b \in \mathbb{N}^*$, is easy to see that $gcd(a, b)$(greatest commom divisor) is a lower bound for $\{a, b\}$. Note that $gcd(2, 3) = 1$, so the unique lower bound of $\{2, 3\}$ is 1. Thus, this poset has no separable smallest element.*

On the other hand, if the order $\leq$ is total, then $\langle A, \leq, <, \perp\rangle$ has separable smallest element.

Definition 2.3 *An **i-Distance Valuation** (IDV) is a structure $\langle A, \leq, R, \perp\rangle$, where R is a semi-auxiliary relation to $\leq$ and $\langle A, \leq, R, \perp\rangle$ is a d-directed poset with separable smallest element.*

Example 2.2 *If $\langle A, \leq, \perp\rangle$ is a totally ordered set, then the structure $\langle A, \leq, <, \perp\rangle$ is a IDV. A very natural IDV is $\langle [0, +\infty), \leq, <, 0\rangle$, where $\leq$ is the usual order of real numbers. This structure is practically the valuation structure for usual metrics (lacking only on the addition operation).*

Definition 2.4 *Let M be a nonempty set and $\mathcal{V} = \langle A, \leq, R, \perp\rangle$ an IDV. A function $d : M \times M \longrightarrow A$ is called **i-metric $\mathcal{V}$-valued** (or with respect to $\mathcal{V}$) whenever:*

1. *$d(a, b) = \perp$ if, and only if, $a = b$;*
2. *$d(a, b) = d(b, a)$, for all $a, b \in M$;*
3. *If $d(a, b)R\varepsilon$, for some $\varepsilon \in A$ with $\perp R\varepsilon$, then exists $\delta \in A$, with $\perp R\delta$, such that $d(b, c)R\delta \Rightarrow d(a, c)R\varepsilon$.*

*In this case, the triplet $(M, d, \mathcal{V})$ is called **i-metric space**.*

The first two conditions of i-metric are easily recognizable as generalizations of the usual conditions of metrics. The third one, which is the "triangle inequality" seems strange, but in section 3, about topology, it will be justified.

In [8], where another distance generalization is introduced, the author stated that the minimum structure necessary to generalize the valuation space of distance has to be able to encompasses the triangle inequality, i.e., an order and a binary operation (the sum) are necessary. Nevertheless, our IDV does not have a binary operation. In section 5 of the metric on fuzzy numbers of the interval type, we explain the reason for this.

The next theorem shows that the usual metrics are i-metrics.

Theorem 2.1 *Let d be an usual metric on M. The function $d_i : M \times M \longrightarrow [0, +\infty)$, defined by $d_i(a, b) = d(a, b)$ is an i-metric $\mathcal{V}$-valued, where $\mathcal{V} = \langle [0, +\infty), \leq, <, 0\rangle$.*

Proof: Let $\langle M, d\rangle$ be an usual metric space. The structure $\mathcal{V} = \langle \mathbb{R}^+, \leq, <, \{0\}\rangle$ is trivially an IDV. Its immediate that the function d_i satisfies the conditions 1. and 2. of i-metric. For the third condition, suppose that $d_i(a, b) < \varepsilon$, with $\varepsilon > 0$. Take $\delta = \varepsilon - d(a, b) > 0$, so $d_i(b, c) < \delta \Rightarrow d(b, c) < \varepsilon - d(a, b) \Rightarrow d(a, b) + d(b, c) < \varepsilon$, so, from the usual triangle inequality, it follows that $d(a, c) < \varepsilon$, ie, $d_i(a, c) < \varepsilon$.

■

3. i-Metrics and Topology

In this section, we show how the i-metrics, as in the usual case, generate a topology from the concept of open ball.

Definition 3.1 *Let $(M, d, \langle A, \leq, R, \perp\rangle)$ be an i-metric space. Given $a \in M$ and $\varepsilon \in A$ with $\perp R\varepsilon$, the **open ball with center** a **and radius** ε is the set $B(a, \varepsilon) = \{b \in M; d(a, b)R\varepsilon\}$. A set $O \subseteq M$ is called **open**, whenever for every $a \in O$ there is an open ball $B(a, \varepsilon)$, such that $B(a, \varepsilon) \subseteq O$.*

Theorem 3.1 *Let $(M, d, \langle A, \leq, R, \perp\rangle)$ be an i-metric space. The class $\Im(M)$ of the open sets of M is a topology on M.*

Proof: It's enough to prove that $\emptyset, A \in \Im(M)$, if $\{A_\lambda\}_{\lambda\in L} \subseteq \Im(M)$, then $\bigcup_{\lambda\in L} A_\lambda \in \Im(M)$ and if $A, B \in \Im(M)$, then $A \cap B \in \Im(M)$. The first two conditions are immediate. Take $A, B \in \Im(M)$ and $a \in A \cap B$. Since A and B are open sets, there are open balls $B(a, \varepsilon_1)$ and $B(a, \varepsilon_2)$ such that $B(a, \varepsilon_1) \subseteq A$ and $B(a, \varepsilon_2) \subseteq B$. We have $\perp R\varepsilon_1$ and $\perp R\varepsilon_2$, so, as A is a d-directed set with separable smallest element, there is a lower bound $\delta \in A$ for $\{\varepsilon_1, \varepsilon_2\}$ with $\perp R\delta$. Thus, consider the open ball $B(a, \delta)$. Take $b \in B(a, \delta)$, i.e., $d(a, b)R\delta$. Since δ is a lower bound for $\{\varepsilon_1, \varepsilon_2\}$, we have $\delta \leq \varepsilon_1 \Rightarrow d(a, b)R\varepsilon_1 \Rightarrow b \in B(a, \varepsilon_1)$, which implies that $B(a, \delta) \subseteq B(a, \varepsilon_1)$. Similarly, we prove that $B(a, \delta) \subseteq B(a, \varepsilon_2)$. Thus, $B(a, \delta) \subseteq A \cap B$, so $A \cap B \in \Im(M)$.

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In the previous proof, the necessity of the valuation space be a d-directed set with separable smallest element and R be a semi-auxiliary relation becomes clear.

Theorem 3.2 *Let $(M, d, \langle A, \leq, R, \perp\rangle)$ be an i-metric space. Every open ball is an open set.*

Proof: Take $\varepsilon \in A$, with $\perp R\varepsilon$, $a \in M$ and $b \in B(a, \varepsilon)$. Thus, $d(a, b)R\varepsilon$. From the third condition of i-metrics, there is $\delta \in A$, with $\perp R\delta$, such that $d(b, c)R\delta \Rightarrow d(a, c)R\varepsilon$. Consider the open ball $B(b, \delta)$. If $c \in B(b, \delta)$, then $d(b, c)R\delta \Rightarrow d(a, c)R\varepsilon \Leftrightarrow c \in B(a, \varepsilon) \Rightarrow B(b, \delta) \subseteq B(a, \varepsilon)$, meaning that $B(a, \varepsilon)$ is an open set.

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In the previous proof, the triangle inequality of i-metric was justified. It follows directly from the above theorem that the class of open ball is a basis to the i-metric topology.

4. Fuzzy Numbers

In this section we make a brief explanation on fuzzy numbers. For more details see [9].

A **fuzzy number** is a fuzzy set $A : \mathbb{R} \longrightarrow [0,1]$ that satisfies:

1. A is normal (there is $t \in \mathbb{R}$ such that $A(t) = 1$);
2. The support of A — i.e. the set $suppA = \{t \in \mathbb{R};\ A(t) > 0\}$ — is a bounded subset of $\mathbb{R}$;
3. For every $\alpha \in (0,1]$, the α-cut $A_\alpha = \{t \in \mathbb{R};\ A(t) \geq \alpha\}$ is a compact interval of $\mathbb{R}$.

Every real number r can be seen as a fuzzy number, with membership function:

$$\mu_r(t) = \left\{ \begin{array}{l} 1, \text{ if } t = r \\ 0, \text{ if } t \neq r \end{array} \right. .$$

The main type of fuzzy numbers to be used in this paper are the intervals, i.e., considering a compact interval $[a,b]$, we can see it as a fuzzy number with membership function:

$$\mu_{[a,b]}(t) = \left\{ \begin{array}{l} 1, \text{ if } t \in [a,b] \\ 0, \text{ if } t \notin [a,b] \end{array} \right. .$$

We will denote the set of this fuzzy numbers by I_f. If $X \in I_f$, we use the notation $X = [\underline{x}, \overline{x}]$. The set of all elements $X \in I_f$ such that $\underline{x} \geq 0$ will be denoted by I_f^+.

Since these fuzzy numbers are identifield with the intervals, we can use the concepts relatives to the interval mathematics([10]). For example, the Kulisch-Miranker order $\leq_{km}$(see [11]), defined by $X \leq_{km} Y \Leftrightarrow \underline{x} \leq \underline{y}$ and $\overline{x} \leq \overline{y}$, for $X, Y \in I_f$.

5. i-Metric on Fuzzy Numbers(Intervals)

An IDV $\mathcal{V}$ whose the set is a class of fuzzy sets will be called **fuzzy IDV** and an i-metric $\mathcal{V}$-valued will be called **fuzzy i-metric**. Next, we construct a fuzzy IDV for our example of fuzzy i-metric on intervals.

Proposition 5.1 *The binary relation $\ll^*$ in I_f^+ defined by:*

1. $[0,a] \ll^* [0,b] \Leftrightarrow a < b$*;*
2. $[a,b] \ll^* [c,d] \Leftrightarrow a < c$ *and* $b < d$*, where* $a, b, c, d > 0$.

is a semi-auxiliary relation to $\leq_{km}$.

Proof: Straightforward.

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Observation 5.1 *Note that if* $[a,b] \in I_f^+$ *and* $[a,b] \neq [0,0]$*, then* $[0,0] \ll^* [a,b]$.

Theorem 5.1 *The structure* $\langle I_f^+, \leq_{km}, \ll^*, [0,0]\rangle$ *is a fuzzy IDV.*

Proof: We trivially see that this structure is a d-directed set with smallest element. To prove that the smallest element is separable, suppose that $X, Y \in I_f^+ - \{[0,0]\}$. Thus, $\overline{x} > 0$ and $\overline{y} > 0$, so $c = \min\{\overline{x}, \overline{y}\} > 0$, implying that $[0,0] \ll^* [0,c] \in L_{\{X,Y\}}$.

■

This IDV $\omega_{KM} = \langle I_f^+, \leq_{km}, \ll^*, [0,0]\rangle$ will be called **IDV of Kulisch-Miranker**.

Next, we present the concept of interval representation. For more details, see [12] and and [13].

Definition 5.1 *A function* $F : I_f^n \longrightarrow I_f$ *is said to be an* ***interval representation*** *of a function* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *whenever* $f([a_1, b_1] \times \cdots \times [a_n, b_n]) \subseteq F([a_1, b_1], \ldots, [a_n, b_n])$ *(here, we consider the fuzzy numbers in* I_f *as simple intervals, i.e, sets of real numbers and the relation* $\subseteq$ *is the usual inclusion relation on sets) for all* $[a_i, b_i] \in I_f$*; in other words, if* $x \in [a_1, b_1] \times \cdots \times [a_n, b_n]$*, then* $f(x) \in F([a_1, b_1], \ldots, [a_n, b_n])$*. In this case we say that F* ***represents*** f*.*

Example 5.1 *Consider the function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *defined by* $f(x) = x+1$*. The functions* $F, G : I_f \longrightarrow I_f$ *defined by* $F([a,b]) = [a, b+1]$ *and* $G([a,b]) = [a+1, b+1]$ *represent* f*.*

Example 5.2 *Consider the function* $F : I_f \longrightarrow I_f$ *defined by* $F([a,b]) = [a-1, b+1]$*. We have that F represents the real functions* $f(x) = x-1$*,* $g(x) = x$ *and* $h(x) = x+1$*.*

In the example 5.1, we see that a real function can be represented by more than one interval function. Note that the functions F and G in this example satisfy $G([a,b]) \subset F([a,b])$, for all $[a,b] \in \mathbb{IR}$. One can say that the function G is closer to f than F, which motivates the definition.

Definition 5.2 *[12] Let* $F, G : I_f^n \longrightarrow I_f$ *be two interval functions that represent the real function* $f : \mathbb{R} \longrightarrow \mathbb{R}$*. Thus, F is said to be a* ***better interval representantion of*** f *than G, which is denoted by* $G \sqsubseteq F$*, whenever* $F([a_1, b_1] \times \ldots \times [a_n, b_n]) \subseteq G([a_1, b_1] \times \ldots \times [a_n, b_n])$*, for all* $[a_i, b_i] \in \mathbb{IR}$*.*

Definition 5.3 *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a real function without vertical asymptotes (* $\lim\limits_{x \longrightarrow a_-^+} f(x) \neq \pm\infty$*, for all* $a \in \mathbb{R}$*). The interval function* $\widehat{f} : I_f \longrightarrow I_f$ *defined by* $\widehat{f}([a,b]) = [\inf f([a,b]), \sup f([a,b])]$ *is called* ***canonical interval representation*** *(CIR for short) of* f*.*

The condition of f does not have vertical asymptotes ensures that the function $\widehat{f}$ is well defined.

Proposition 5.2 *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a real function without vertical asymptotes. If* $F : I_f \longrightarrow I_f$ *represents* f*, then* $F \sqsubseteq \widehat{f}$*, i.e., the function* $\widehat{f}$ *is the* ***best interval representation*** *of* f*.*

Proof: See [12]. ∎

The next proposition shows that the notion of fuzzy i-metric introduced here does not capture the idea of interval representation to usual Euclidean metric on $\mathbb{R}$.

Theorem 5.2 *There is no fuzzy i-metric $d_i : I_f \times I_f \longrightarrow I_f^+$ $\mathcal{V}$-valued, where $\mathcal{V} = \langle \mathbb{IR}^+, \leq, R, [0,0]\rangle$, that represents the usual Euclidean distance on $\mathbb{R}$.*

Proof: As $d_i([a,b],[a,b]) = [0,0]$, if $a < b$, then exists $x, y \in [a,b]$ such that $d(x,y) = |x-y| > 0$. Thus, $x, y \in [a,b]$, but $d(x,y) \notin d_i([a,b],[a,b]) = [0,0]$. ∎

The canonical interval representation of a real function is the best interval representation of this function. For this reason, we consider that a generalized notion of distance **captures the idea** of interval representation whenever this notion includes the function $D([a,b],[c,d]) = \widehat{d}([a,b],[c,d])$, if $[a,b] \neq [c,d]$ and $D([a,b],[c,d]) = [0,0]$, if $[a,b] = [c,d]$ (here, d is the usual Euclidean distance on $\mathbb{R}$). This function differs from $\widehat{d}$ only on the value of $D(X,X)$, where X is a non-degenerate interval.

Theorem 5.3 *Given $X, Y \in I_f$, consider the set $D_{XY} = \{d(x,y); x \in X \text{ e } y \in Y\} = [\min D_{XY}, \max D_{XY}] \in I_f^+$. This element of I_f represents all the possible values of the (usual Euclidean) distances betweeen elements of the intervals $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$. The function $d_{KM} : I_f \times I_f \longrightarrow I_f^+$ defined by:*

$$d_{KM}(X,Y) = \begin{cases} [0,0] & , \text{ if } X = Y \\ D_{XY} & , \text{ if } X \neq Y \end{cases} \quad (1)$$

is a fuzzy i-metric ω_{KM}-valued.

Proof: If $X = Y$, then $d_{KM}(X,Y) = [0,0]$. Suppose that $X \neq Y$. Thus, there are $x \in X$ and $y \in Y$ with $x \neq y$, so $d(x,y) > 0$ and, consequently, $\max D_{XY} > 0$, meaning that $d_{KM}(X,Y) \neq [0,0]$. So, The first condition is hold.

Since d is an usual metric, the second condition is immediate.

Finally, suppose that $d_{KM}(X,Y) \ll^* \Sigma = [\underline{\varepsilon}, \overline{\varepsilon}]$, with $[0,0] \ll^* \Sigma$. If $X = Y$, the result holds trivially. Thus, suppose that $X \neq Y$, so $\max D_{XY} < \overline{\varepsilon}$. First, consider $\underline{y} < \overline{y}$. In this case, take $\Delta = [0, \frac{\overline{y} - \underline{y}}{2}]$. Note that $[0,0] \ll^* \Delta$. If $z \in \mathbb{R}$, then $d(z, \underline{y}) \geq \frac{\overline{y} - \underline{y}}{2}$ or $d(z, \overline{y}) \geq \frac{\overline{y} - \underline{y}}{2}$, so for every interval Z, we have $\max D_{YZ} \geq \frac{\overline{y} - \underline{y}}{2}$. Thus, if $Z \neq Y$, then $d_{KM}(Y,Z) \not\ll^* \Delta$. So, $d_{KM}(Y,Z) \ll^* \Delta \Rightarrow Y = Z \Rightarrow d_{KM}(X,Z) = d_{KM}(X,Y) \ll^* \Sigma$.

Consider the case $Y = [y, y]$. Define $\Delta = [0, \overline{\varepsilon} - \sup D_{XY}]$ and suppose that $d_{KM}(Y,Z) \ll^* \Delta$. If $Y = Z$, then the result holds trivially. If $Y \neq Z$, then $\min D_{YZ} = 0 \Rightarrow y \in Z \Rightarrow \min D_{XZ} \leq \min D_{XY}$ and $\max D_{YZ} < \overline{\varepsilon} - \max D_{XY} \Rightarrow \max D_{XY} + \max D_{YZ} < \overline{\varepsilon}$. Define $D = \{d(x,y) + d(y,z);\ x \in X,\ y \in Y,\ z \in Z\}$. Note that $D \subseteq D_{XY} + D_{YZ} \Rightarrow \max D \leq \max(D_{XY} + D_{YZ}) = \max D_{XY} + \max D_{YZ} < \overline{\varepsilon}$. From the usual triangle inequality, it follows that $\max D_{XZ} \leq \max D < \overline{\varepsilon}$. So, $d_{KM}(X,Z) \ll^* \Sigma$. ∎

The function d_{KM} will be called KM-metric.

Considering the usual sum of intervals (see [10]) and the Kulisch-Miranker order, this function d_{KM} does not satisfies the usual triangle inequality. In fact, take $X = [0,1]$, $Y = [1,2]$ and $Z = [2,3]$. Thus, $d_{KM}(X,Y) = [0,2]$, $d_{KM}(X,Z) = [1,3]$ and $d_{KM}(Y,Z) = [0,2]$, so $d_{KM}(X,Z) \not\leq_{KM} d_{KM}(X,Y) + d_{KM}(Y,Z)$. This fact justifies the triangle inequality of i-metric, since the function d_{KM} is a very natural generalization of the Euclidean distance on $\mathbb{R}$.

The next theorem presents the characterization of the i-metric d_{KM}.

Theorem 5.4 *Given $X, Y \in I_f$, we have:*

$$d_{\mathrm{km}}(X,Y) = \begin{cases} [0,0] & , \text{ if } X = Y \\ [d(\overline{x}, \underline{y}), d(\underline{x}, \overline{y})] & , \text{ if } \overline{x} < \underline{y} \\ [d(\underline{x}, \overline{y}), d(\overline{x}, \underline{y})] & , \text{ if } \overline{y} < \underline{x} \\ [0, d(\underline{x}, \overline{y})] & , \text{ if } X <_{\mathrm{km}} Y \\ & \text{ and } X \cap Y \neq \emptyset \\ [0, d(\overline{x}, \underline{y})] & , \text{ if } Y <_{\mathrm{km}} X \\ & \text{ and } X \cap Y \neq \emptyset \\ [0, (d(\underline{x}, \overline{y})) \vee (d(\overline{x}, \underline{y}))] & , \text{ if } X \neq Y \text{ and} \\ & (X \subset Y \text{ or } Y \subset X) \end{cases}$$

Proof: Immediate. ∎

6. The topology Generated by d_{KM}

In this section, we present some properties of the topology generated by d_{KM}, which will be denoted by $\Im_{KM}$. For example, this topology is Hausdorff and regular. In the end, we prove that this topology is not metrizable. We start with the lemma below.

Lemma 6.1 *If $X = [\underline{x}, \overline{x}]$ with $\underline{x} < \overline{x}$, then $\{X\}$ is an open set of $\Im_{KM}$.*

Proof: Consider $X \in I_f$, with $\underline{x} < \overline{x}$. Take $\Delta = [0, \frac{\overline{x} - \underline{x}}{2}]$. Note that $[0,0] \ll^* \Delta$ and if $x \neq y$, then $\sup D_{XY} \geq \frac{\overline{x} - \underline{x}}{2} \Rightarrow d_{KM}(X,Y) \not\ll^* \Delta$. Thus, $d_{KM}(X,Y) \ll^* \Delta \Rightarrow X = Y$, that is, the open ball(which is an open set of $\Im_{KM}$), $B(X, \Delta)$ is equal to $\{X\}$.

∎

This lemma shows that the topology $\Im_{KM}$ is "almost trivial", since every singleton $\{X\}$, where $X = [\underline{x}, \overline{x}]$ with $\underline{x} < \overline{x}$, is an open set. However, this topology is not trivial. The singleton $\{[a,a]\}$ is not an open set for all $a \in \mathbb{R}$. In fact, given $[0,0] \ll^* [\underline{\varepsilon}, \overline{\varepsilon}]$, we have $\overline{\varepsilon} > 0$. Thus, take $X = [a, a + \frac{\overline{\varepsilon}}{2}]$. Note that $d_{KM}([a,a], X) = [0, \frac{\overline{\varepsilon}}{2}]) \ll^* [\underline{\varepsilon}, \overline{\varepsilon}]$, so, $X \in B([a,a], [\underline{\varepsilon}, \overline{\varepsilon}])$. It follows that every open ball with center $[a,a]$ has some other element from $\mathbb{IR}$ than $[a,a]$. Another interesting fact about $\Im_{KM}$ is that for every $[a,a] \in I_f$, there is an open ball $B_1 = B([a,a], \Delta)$ such that the unique degenerated element(that is, $[x,x] \in If$) in B_1 is $[a,a]$. In fact, just take $\Delta = [0, \varepsilon]$, with $\varepsilon > 0$.

Proposition 6.1 *The topology $\Im_{KM}$ is Hausdorff.*

Proof: Take $X, Y \in I_f$. We must prove that there is two disjoint open sets A and B such that $X \in A$ and $Y \in B$. If X and Y are non-degenerated elements of I_f, then take $A = \{X\}$ and $B = \{Y\}$.

Now, consider the case $X = [x,x]$ and $Y = [\underline{y}, \overline{y}]$, with $\underline{y} < \overline{y}$. Take $0 < r < \sup_{y \in Y} |x - y|$ and consider the open ball $B = B([x,x],[0,r])$. Note that $d_{KM}([x,x], [\underline{y}, \overline{y}]) = [\inf_{y \in Y} |x-y|, \sup_{y \in Y} |x-y|]$, so, as $\sup_{y \in Y} |x-y| > r$, it follows that $[\underline{y}, \overline{y}] \notin B$. Thus, B and $\{[\underline{y}, \overline{y}]\}$ are the two disjoint open sets we were looking for.

Finally, consider the case $X = [x,x]$ and $Y = [y,y]$. Take $r = |x-y| > 0$ and the open balls $B_1 = B([x,x],[0,\frac{r}{2}]$ and $B_2 = B([y,y],[0,\frac{r}{2}]$. The unique degenerate interval in B_1 is $[x,x]$ and in B_2 is $[y,y]$. Take $A = [\underline{a}, \overline{a}]$, with $\underline{a} < \overline{a}$ and suppose that $A \in B_1 \cap B_2$. Thus, $\sup_{a \in A} |x - a| < \dfrac{r}{2}$ and $\sup_{a \in A} |y - a| < \dfrac{r}{2}$. Consider $a_1, a_2 \in A$ such that $|x - a_1| = \sup_{a \in A} |x-a|$ and $|y - a_2| = \sup_{a \in A} |y - a|$. So, $|x - a_1| < \frac{r}{2}$ and $|y - a_2| < \frac{r}{2}$, which implies(by the usual triangle inequality) that $|x-y| \leq |x - a_1| + |y - a_1| \leq |x - a_1| + |y - a_2| < \frac{r}{2} + \frac{r}{2} = r$, i.e., $|x-y| < r$, which is a contradiction, so, $B_1 \cap B_2 = \emptyset$.

∎

Because of the above theorem we can ask ourselves if the topology $\Im_{KM}$ is metrizable, i.e., can it be generated by an usual metric, since every metrizable topology is Hausdorff. To investigate this we can see if $\Im_{KM}$ has other properties of the metrizable topologies. It is a well-known fact that every metrizable topology is regular, that is, the topology is Hausdorff and for every point x and closed set F such that $x \notin F$ we can find disjoint open sets A and B such that $x \in A$ and $F \subseteq B$. To prove that this topology is regular we will use the lemma below, whose proof can be found in [5].

Lemma 6.2 *A topological space (M, τ) (or the topology τ) is regular if, and only if, for every $a \in M$ and for every neighborhood O of a, there is a neighborhood V of a such that $a \in V \subseteq \overline{V} \subseteq O$ ($\overline{V}$ represents the closure of V).*

Lemma 6.3 *The open balls $B([a,a],[0,r])$, with $r > 0$ are closed sets with respect to $\Im_{KM}$.*

Proof: In fact, we must prove that the complementary set of $B = B([a,a],[0,r])$ is an open set of $\Im_{KM}$. If $[\underline{x}, \overline{x}] \notin B$, with $\underline{x} < \overline{x}$, then $\{[\underline{x}, \overline{x}]\}$ is a neighborhood of $[\underline{x}, \overline{x}]$ in the complement of B. For $[x,x] \notin B$, we have $|x - a| = r_1 > 0$. Thus, take the open ball $B_1 = B([x,x],[0,r_1])$. We must prove that this open ball is a subset of the complement of B. Note that the only degenerate interval in B_1 is $[x,x]$, which is not in B. Now, consider $X = [\underline{x}, \overline{x}] \in B_1$, with $\underline{x} < \overline{x}$. We must have $\underline{x} \leq x \leq \overline{x}$ and $\sup D_{X[x,x]} < r_1$. Since $\sup D_{X[x,x]} = \max\{x - \underline{x}, \overline{x} - x\}$, it follows that $x - \underline{x} < r_1$ and $\overline{x} - x < r_1$. Thus, given $z \in [\underline{x}, \overline{x}]$, we have $|z - x| < r_1$, so $a \notin [\underline{x}, \overline{x}]$ and then $[\underline{x}, \overline{x}] \notin B$.

∎

Theorem 6.1 *The topology $\Im_{KM}$ is regular.*

Proof: If $X = [\underline{x}, \overline{x}]$, with $\underline{x} < \overline{x}$, then $V = \{X\}$ is an open and closed set, then $X \in V \subseteq \overline{V} \subset U$ for all open set U such that $X \in U$. If $X = [x,x]$ and U is an open set such that $X \in U$, then there is an open ball $B = B(X,[0,r])$ such that $B \subseteq U$. Since B is a closed set, we have $B = \overline{B}$, so $X \subseteq B \subseteq \overline{B} \subseteq U$. Thus, it follows from the lemma above that $\Im_{KM}$ is regular.

∎

Thus, the topology $\Im_{KM}$ is regular, that is, it has other property of metrizable topologies. To investigate if $\Im_{KM}$ is a metrizable topology, we use the below theorem, which gives a characterization of metrizable topologies, whose the proof can be found in [5].

Theorem 6.2 (Nagata-Smirnov theorem) *A topology τ is metrizable if, and only if, is regular(and then Hausdorff) and has a basis that can be decomposed into an at most countable collection of locally finite families.*

A familly $\mathcal{A}$ of subsets from a topological space (M, τ) is locally finite if for every $x \in M$ there is a neighborhood of x that intersects only finitely many sets of the family.

Since the topology $\Im_{KM}$ is regular, to prove that it is non metrizable we must prove that it has no basis as in the Nagata-Smirnov theorem.

Theorem 6.3 *The topology $\Im_{KM}$ is not metrizable.*

Proof: Suppose that $\Im_{KM}$ is metrizable. From the Nagata-Smirnov theorem, it follows that $\Im_{KM}$ has a basis $\mathcal{B}$ that can be decomposed into an at most countable collection of locally finite families. Since every singleton set $\{[\underline{x}, \overline{x}]\}$, with $\underline{x} < \overline{x}$, is an open set of $\Im_{KM}$, so every basis of $\Im_{KM}$ must contain all of this kind of singleton. Denote by N the class of this singletons. Thus, we have $N \subseteq \mathcal{B}$. As $\mathcal{B}$ can be decomposed into an at most countable collection of locally finite families, so can N. In fact, if $\mathcal{B} = \bigcup_{n\in\mathbb{N}} B_n$ is that decomposition for $\mathcal{B}$, then $N = \bigcup_{n\in\mathbb{N}}(N \cap B_n)$ is the decomposition for $\mathcal{N}$. We will use the notation $L_n = N \cap B_n$. Take $x \in \mathbb{R}$ and consider the degenerate interval $[x, x]$ of $\mathbb{IR}$. By the initial hypotheses, there is an open set $V \in \Im_{KM}$ such that $[x, x] \in V$ and V intersects a finite quantity of sets in L_n, for every $n \in \mathbb{N}$. Since V is an open set, there is an open ball B_x with center $[x, x]$ such that $B_x \subseteq V$. Suppose, without loss of generality, that $B_x = B([x, x], [0, r])$, with $r > 0$. Thus, for every $n \in \mathbb{N}$, B_x intersects only finetely many sets in L_n, that is, there is a finite quantity of intervals $[\underline{x}, \overline{x}]$, with $\underline{x} < \overline{x}$, belonging to B_x such that $\{[\underline{x}, \overline{x}]\} \in L_n$. Consider the intervals $X_s = [x, x + s]$, with $0 < s < r$. Note that $X_s \in B_x$, for all $s \in (0, r)$, because $d_{km}([x, x], X_s) = [0, s] \ll^* [0, r]$. Define $N_x = \{X_s;\ 0 < s < r\}$. We have $N_x \subseteq N = \bigcup_{n\in\mathbb{N}} L_n$, so $N_x = \bigcup_{n\in\mathbb{N}}(N_x \cap L_n)$. Since for every $n \in \mathbb{N}$, $N_x \cap L_n$ is a finite set, so N_x is a union of a countable quantity of finite sets, so N_x is a countable set, which is an absurd because N_x has the same cardinality of $(0, r)$. Thus, $\Im_{KM}$ is a non-metrizable topology.

■

7. Final Remarks

In this paper we presented a new concept of generalized metric based on the modification of the codomain of the distance function and show how this notion generates a topology quite naturally (like the topology generated by an usual metric). From this notion, we presented an example of i-metric with fuzzy valuation. In [4] the authors showed that the topology generated by the fuzzy metric proposed by George and Veeramani was metrizable. The authors of [4] commented that this was a good result, because with it the topology would have several properties that are typical of metrizable topologies. In fact, this may be interesting, however, from the topological point of view this makes the notion of fuzzy metric by George and Veeramani unnecessary, since the topology can be generated by an usual metric. In this paper, we show that our example of fuzzy i-metric generates a Hausdorff and regular topology, which is metrizable. This justifies our proposal of generalized metric.

In future works, we intend to find new examples of i-metric whose codomain classes formed by other types of fuzzy sets, such as the triangular fuzzy numbers, or trapezoidal. Also, since the fuzzy numbers are defined by its α-cuts, which are intervals, we can use the KM-metric to define a fuzzy number from the distance between the α-cuts of two fuzzy numbers, which provides a fuzzy number as the distance between fuzzy numbers. In addition, we can study i-metrics with other valuations, as sets of strings or functions.

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