Strong Law of Large Numbers for α-Mixing Sequences

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Abstract—*For independent identically distributed random variables, the Marcinkiewicz strong law of large numbers is that sppose $EX_n = 0$, Then $n^{-1/p}S_n \to 0, n \to \infty, a.s.$ if and only if $E|X_1|^p < \infty$. Let $\{X_n, n \ge 1\}$ be an identically distributed α -mixing sequence of random variables, in this paper , the Marcinkiewicz's strong law of large numbers for $\{X_n, n \ge 1\}$ is discussed, by using Wang Xiaoming's Borell-Cantelli Lemma (Wang X. M.1997), we have the following equiverlences, $\forall \varepsilon > 0$*

$$2^{-nr}(S_{2^n+2^{n-1}} - S_{2^n}) \to 0, n \to \infty, a.s. \Leftrightarrow \sum_{n=1}^{\infty} P(A_n^{(1)}) < \infty,$$

$$2^{-nr}(S_{2^{n+1}} - S_{2^n+2^{n-1}}) \to 0, n \to \infty, a.s. \Leftrightarrow \sum_{n=1}^{\infty} P(A_n^{(2)}) < \infty,$$

$$2^{-nr} \max_{2^n \le j \le 2^n+2^{n-1}} |S_j - S_{2^n}| \to 0, \quad n \to \infty, \quad a.s. \Leftrightarrow \sum_{n=1}^{\infty} P(B_n^{(1)}) < \infty,$$

$$2^{-nr} \max_{2^n+2^{n-1} \le j \le 2^{n+1}} |S_j - S_{2^n+2^{n-1}}| \to 0, n \to \infty, a.s. \Leftrightarrow \sum_{n=1}^{\infty} P(B_n^{(2)}) < \infty .$$

By use of Herrndorf's maximal inequality (Herrndorf N. 1983), necessary conditions for Marcinkiewicz's strong law of large numbers are obtained, which require low mixing speed. As a consequence, by use of Shao Qiman's result(1995), we obtain the Marcinkiewicz's strong law of large numbers for ρ -mixing sequence of random variables.

Keywords- α -mixing; strong law of large numbers; necessary conditions; strong convergence. AMS 2000 subject classication: 60F15, 60G50.

I. INTRODUCTION AND MAIN RESULTS

Let $\{X_n, n \ge 1\}$ be a sequence of random variables,

let $S_n = \sum_{i=1}^{n} X_i$, $n \ge 1; S_0 = 0$,

$$\mathscr{F}_1^k = \sigma(X_i, 1 \le i \le k),$$

$$\mathscr{F}_k^\infty = \sigma(X_i, i \ge k).$$

For $n \ge 0$, let

$$\alpha(n) @ \sup_{k \ge 1} \sup\{|P(AB) - P(A)P(B)|, \quad A \in \mathscr{F}_1^k, B \in \mathscr{F}_{k+n}^\infty\};$$

$$\varphi(n) @ \sup_{k \ge 1} \sup\{|P(B/A) - P(B)|, \quad A \in \mathscr{F}_1^k, B \in \mathscr{F}_{k+n}^\infty, P(A) > 0\};$$

$$\rho(n) @ \sup_{k \ge 1} \sup\{|EXY - EXEY| : \quad X \in L_2(\mathscr{F}_1^k), Y \in L_2(\mathscr{F}_{k+n}^\infty)\}.$$

If $\alpha(n) \to 0$, $\varphi(n) \to 0$, $\rho(n) \to 0$, $n \to \infty$ respectively, the $\{X_n, n \ge 1\}$ is called α -mixing, φ -mixing, ρ - mixing respectively.

For i.i.d random variables, the Marcinkiewicz strong law of large numbers is that

Theorem A . sppose $EX_n = 0$, Then

$$n^{-\frac{1}{p}} S_n \to 0, n \to \infty, a.s. \tag{1}$$

if and only if $E|X_1|^p < \infty$.

Lately there has been a great amount of work on strong law of large numbers for dependent random variables, such as the discussion by Wang Xiaoming(1997) Xue Liugen(1994) on φ -mixing sequences, the discussion by Kuczmaszewska A.(2005), Shao Qiman(1995) on ρ -mixing sequences, and the discussion by Meng Y. J., Lin Z.Y.(2010), Bryc W. , Smolenski W.(1993), Peligrad M., Gut A. Yang S. C. (1998), Kuczmaszewska (2008) on $\tilde{\rho}$-mixing sequences. In this paper, we obtain the necessary conditions for strong law of large numbers similar to that of theorem A for α -mixing sequences.

Theorem 1. Let $\{X_n, n \ge 1\}$ be an identically distributed α -mixing sequence of random variables, assume that

$$\sum_{n=1}^{\infty} \alpha(2^n) < \infty \tag{2}$$

$$\varphi(1) < 1 \tag{3}$$

If for some $r > 0$,

$$\lim_{n \to \infty} n^{-r} S_n = 0 \tag{4}$$

Then

$$E|X_1|^{\frac{1}{r}} < \infty \tag{5}$$

Theorem 2. Let $\{X_n, n \ge 1\}$ be an identically distributed ρ -mixing sequence of random variables with

$EX_n = 0$, assume that $1 \le p < 2$, $\varphi(1) < 1$, $\sum_{n=1}^{\infty} \rho(2^n) < \infty$. Then

$$\lim_{n\to\infty} n^{-1/p} S_n = 0. \quad a.s.$$

if and only if

$$E|X_1|^p < \infty.$$

2. PROOF OF THE THEOREMS

To prove our theorem, we need the following lemmas.

Lemma 1. (Wang Xiaoming 1997) Let $\{X_n, n \ge 1\}$ be an α-mixing sequence of random variables, assume that $A_n \in \mathscr{F}_{u_i}^{v_i}$, where u_i and v_i are positive integers satisfying $u_i \le v_i \le u_{i+1} \le v_{i+1}, i = 1, 2, L$,

if $\sum_{i=1}^{\infty} \alpha(u_{i+1} - v_i) < \infty$, then

$$P(A_n, \text{i.o.}) = 1 \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} P(A_n) = \infty.$$

Lemma 2. Suppose that $\{X_n, n \ge 1\}$ satisfies (2), then for r > 0,

$$2^{-nr}(S_{2^n+2^{n-1}} - S_{2^n}) \to 0, n \to \infty, a.s. \qquad (6)$$

$$\Longleftrightarrow \quad \sum_{n=1}^{\infty} P(A_n^{(1)}) < \infty, \quad \forall \varepsilon > 0; \qquad (7)$$

$$2^{-nr}(S_{2^{n+1}} - S_{2^n+2^{n-1}}) \to 0, n \to \infty, a.s. \qquad (8)$$

$$\Longleftrightarrow \quad \sum_{n=1}^{\infty} P(A_n^{(2)}) < \infty, \quad \forall \varepsilon > 0. \qquad (9)$$

Where

$$A_n^{(1)} @ A_n^{(1)}(\varepsilon, r)$$
$$@\{|S_{2^n+2^{n-1}} - S_{2^n}| \ge \varepsilon 2^{nr}\}, n \ge 1$$
$$A_n^{(2)} @ A_n^{(2)}(\varepsilon, r)$$
$$@\{|S_{2^{n+1}} - S_{2^n+2^{n-1}}| \ge \varepsilon 2^{nr}\}, n \ge 1$$

Proof. Clearly, $A_n^{(1)} \in \mathscr{F}_{2^n+1}^{3\times 2^{n-1}}$, then $u_n = 2^n + 1$, $v_n = 3 \times 2^{n-1}$, by (2) we have

$$\sum_{n=1}^{\infty} \alpha(u_{n+1} - v_n) = \sum_{n=1}^{\infty} \alpha(2^{n-1} + 1)$$
$$\le \sum_{n=1}^{\infty} \alpha(2^{n-1}) < \infty$$

By lemma 1, we have

$$(7) \Longleftrightarrow P(A_n^{(1)}, \text{i.o.}) = 0 \Longleftrightarrow (6).$$

Now, we prove $(8) \Longleftrightarrow (9)$.

Obviously, $A_n^{(1)} \in \mathscr{F}_{3\times 2^{n-1}+1}^{2^{n+1}}$, then $u_n = 3 \times 2^{n-1} + 1$, $v_n = 2^{n+1}$, by (2) we have

$$\sum_{n=1}^{\infty} \alpha(u_{n+1} - v_n) = \sum_{n=1}^{\infty} \alpha(2^n + 1)$$
$$\le \sum_{n=1}^{\infty} \alpha(2^n) < \infty$$

By lemma 1, we have

$$(9) \Longleftrightarrow P(A_n^{(2)}, \text{i.o.}) = 0 \Longleftrightarrow (8).$$

The proof of lemma 2 is complete.

Lemma 3. (Herrndorf N. 1983) Let $\{X_n, n \ge 1\}$ be an arbitrary sequences of random variables. Suppose that q is a positive integer. Then for any $a > 0$, every positive integer $s \ge q + 1$, and nonnegative integer m,

$$\left(1 - \varphi(q) - \max_{q \le j \le s} P(|S_{m+s} - S_{m+j}| \ge a)\right)$$
$$\cdot P\left(\max_{j \le s} |S_{m+j} - S_m| \ge 3a\right)$$
$$\le P(|S_{m+s} - S_m| \ge a) + P\left((q-1)\max_{j \le s} |X_{m+j}| \ge a\right)$$

Lemma 4. Suppose that $\{X_n, n \ge 1\}$ satisfies (2), then for r > 0,

$$2^{-nr} \max_{2^n \le j \le 2^n + 2^{n-1}} |S_j - S_{2^n}| \to 0, \quad n \to \infty, \quad a.s. \qquad (10)$$

$$\Longleftrightarrow \quad \sum_{n=1}^{\infty} P(B_n^{(1)}) < \infty, \quad \forall \varepsilon > 0; \qquad (11)$$

$$2^{-nr} \max_{2^n + 2^{n-1} \le j \le 2^{n+1}} |S_j - S_{2^n+2^{n-1}}| \to 0, \quad n \to \infty, \quad a.s. \qquad (12)$$

$$\Longleftrightarrow \quad \sum_{n=1}^{\infty} P(B_n^{(2)}) < \infty, \quad \forall \varepsilon > 0; \qquad (13)$$

Where

$$B_n^{(1)} @ B_n^{(1)}(\varepsilon, r)$$
$$@\{\max_{2^n \le j \le 2^n + 2^{n-1}} |S_j - S_{2^n}| \ge \varepsilon 2^{nr}\}, n \ge 1;$$
$$B_n^{(2)} @ B_n^{(2)}(\varepsilon, r)$$
$$@\{\max_{2^n + 2^{n-1} \le j \le 2^{n+1}} |S_j - S_{2^n+2^{n-1}}| \ge \varepsilon 2^{nr}\}, n \ge 1.$$

Proof. Clearly, $B_n^{(1)} \in \mathscr{F}_{2^n+1}^{3\times 2^{n-1}}$, then $u_n = 2^n + 1$, $v_n = 3 \times 2^{n-1}$, by (2) we have

$$\sum_{n=1}^{\infty} \alpha(u_{n+1} - v_n) = \sum_{n=1}^{\infty} \alpha(2^{n-1} + 1)$$
$$\le \sum_{n=1}^{\infty} \alpha(2^{n-1}) < \infty$$

By lemma 1, we have

$$(11) \Longleftrightarrow P(B_n^{(1)}, \text{i.o.}) = 0 \Longleftrightarrow (10).$$

Now, we prove $(12) \Longleftrightarrow (13)$.

Obviously, $B_n^{(1)} \in \mathscr{F}_{3\times 2^{n-1}+1}^{2^{n+1}}$, then $u_n = 3 \times 2^{n-1} + 1$, $v_n = 2^{n+1}$, by (2) we have

$$\sum_{n=1}^{\infty}\alpha(u_{n+1}-v_n)=\sum_{n=1}^{\infty}\alpha(2^n+1)$$
$$\le\sum_{n=1}^{\infty}\alpha(2^n)<\infty$$

By lemma 1, we have

$(13)\Longleftrightarrow P(A_n^{(2)},\text{i.o.})=0\Longleftrightarrow(12)$.

The proof of lemma 3 is complete.

Lemma 5. (Shao Q. M. 1995) Suppose that $1/2<\alpha\le1, p\alpha\ge1$. Let $\{X_n, n\ge1\}$ be an identically distributed ρ-mixing sequence of random variables with $EX_n=0$ and $E|X_n|^p<\infty$. Assume that

$$\sum_{n=1}^{\infty}\rho^{2/r}(2^n)<\infty.$$

Where $r=2$ if $1\le p<2$ and $r>p$ if $p\ge2$, Then for all $\varepsilon>0$,

$$\sum_{n=1}^{\infty}n^{p\alpha-2}P(\max_{i\le n}|S_i|\ge\varepsilon n^{\alpha})<\infty.$$

Proof of theorem 1. By (4), we have

$$n^{-r}S_n\xrightarrow{P}0, n\to\infty. \qquad (14)$$

$$2^{-nr}\left(S_{2^n+2^{n-1}}-S_{2^n}\right)=2^{-nr}S_{3\times2^{n-1}}-2^{-nr}S_{2^n}$$
$$=\left(\frac{3}{2}\right)^{-r}\left(3\times2^{n-1}\right)^{-r}S_{3\times2^{n-1}}-2^{-nr}S_{2^n}\to0,$$
$$n\to\infty, a.s$$

That is $(4)\Rightarrow(6)$.

By (14), we have

$$2^{-nr}(S_{2^{n+1}}-S_{2^n+2^{n-1}})$$
$$=2^{-nr}S_{2^{n+1}}-2^{-nr}S_{2^n+2^{n-1}}$$
$$=2^r\left(2^{n+1}\right)^{-r}S_{2^{n+1}}-\left(\frac{3}{2}\right)^{-r}\left(3\times2^{n-1}\right)^{-r}S_{3\times2^{n-1}}$$
$$\to0,\quad n\to\infty, a.s$$

That is $(4)\Rightarrow(8)$.

By lemma 2 we get $(6)+(8)\Rightarrow(7)+(9)$. In the following, we show that $(14)+(7)\Rightarrow(11)$.

Put $m=2^n, s=2^{n-1}, q=1, a=\varepsilon2^{nr}$, by lemma 3 we obtain

$$\left(1-\varphi(q)-\max_{q\le j\le s}P(|S_{m+s}-S_{m+j}|\ge a)\right)$$
$$\cdot P(\max_{j\le s}|S_{m+j}-S_m|\ge3a)$$
$$\le P(|S_{m+s}-S_m|\ge a)+P\left((q-1)\max_{j\le s}|X_{m+j}|\ge a\right)$$

By (3), we get $1-\varphi(1)>0$,

$$\max_{1\le j\le2^{n-1}}P(|S_{2^n+2^{n-1}}-S_{2^n+j}|\ge\varepsilon2^{nr})$$
$$\le\max_{1\le j\le2^{n-1}}\left(P(|S_{2^n+2^{n-1}}|\ge\frac{\varepsilon}{2}2^{nr})+P(|S_{2^n+j}|\ge\frac{\varepsilon}{2}2^{nr})\right)$$
$$\le2\max_{1\le j\le2^{n-1}}P(|S_{2^n+j}|\ge\frac{\varepsilon}{2}2^{nr})$$

Combining $1-\varphi(1)>0$ with (14) yields that there exists a positive integer N_1 such that for $n>N_1$,

$$2\max_{1\le j\le2^{n-1}}P(|S_{2^n+j}|\ge\frac{\varepsilon}{2}2^{nr})\le\frac{1}{2}\left(1-\phi(1)\right).$$

Hence,

$$\max_{1\le j\le2^{n-1}}P(|S_{2^n+2^{n-1}}-S_{2^n+j}|\ge\varepsilon2^{nr})\le\frac{1}{2}\left(1-\phi(1)\right),$$

By (15), we obtain

$$\max_{j\le2^{n-1}}P(|S_{2^n+j}-S_{2^n}|\ge3\varepsilon2^{nr})$$
$$\le2\left(1-\phi(1)\right)^{-1}P(|S_{2^n+2^{n-1}}-S_{2^n}|\ge\varepsilon2^{nr}).$$

So (11) follows from (7) immediately. In the following, we show that

$(11)+(13)\Rightarrow$

$$P(\max_{2^n<j\le2^{n+1}}|S_{2j}-S_{2^n}|\ge\varepsilon2^{nr})\le\infty,\quad\forall\varepsilon>0 \qquad (16)$$

Since

$$\left\{\max_{2^n<j\le2^{n+1}}|S_j-S_{2^n}|\ge\varepsilon2^{nr}\right\}$$
$$=\left\{\max_{2^n<j\le2^n+2^{n-1}}|S_j-S_{2^n}|\ge\varepsilon2^{nr}\right\}$$
$$\mathrm{U}\left\{\max_{2^n+2^{n-1}<j\le2^{n+1}}|S_j-S_{2^n}|\ge\varepsilon2^{nr}\right\}$$
$$\subset B_n^{(1)}(\varepsilon,r)$$
$$\mathrm{U}\left\{\max_{2^n+2^{n-1}<j\le2^{n+1}}|S_j-S_{2^n+2^{n-1}}|\ge\frac{\varepsilon}{2}2^{nr}\right\}$$
$$\mathrm{U}\left\{|S_{2^n+2^{n-1}}-S_{2^n}|\ge\frac{\varepsilon}{2}2^{nr}\right\}$$
$$\subset B_n^{(1)}(\varepsilon/2,r)\,\mathrm{U}\,B_n^{(2)}(\varepsilon/2,r),\ \forall\varepsilon>0.$$

So (16) follows from (11) and (13) immediately. Since

$$P(\max_{2^n<j\le2^{n+1}}|X_j|\ge\varepsilon2^{nr})$$
$$=P(\max_{2^n<j\le2^{n+1}}|S_j-S_{j-1}|\ge\varepsilon2^{nr})$$
$$\le P(\max_{2^n<j\le2^{n+1}}|S_j-S_{2^n}|\ge\frac{\varepsilon}{2}2^{nr})$$

$$+P(\max_{2^n<j\le 2^{n+1}} |S_{j-1}-S_{2^n}| \ge \frac{\varepsilon}{2}2^{nr})$$

$$\le 2P(\max_{2^n<j\le 2^{n+1}} |S_j-S_{2^n}| \ge \frac{\varepsilon}{2}2^{nr})$$

So (16) yields that

$$\sum_{n=1}^{\infty} P(\max_{2^n<j\le 2^{n+1}} |X_j| \ge \varepsilon 2^{nr}) < \infty, \forall \varepsilon > 0 \qquad (17)$$

It follows from (17) that

$$\sum_{n=1}^{\infty} P(\max_{j\le 2^n} |X_j| \ge \varepsilon 2^{nr}) < \infty, \forall \varepsilon > 0. \qquad (18)$$

It is easy to see that

$$\sum_{n=2^m+1}^{2^{m+1}} \frac{1}{n} P(\max_{j\le n} |X_j| \ge \varepsilon n^r)$$

$$\le \sum_{n=2^m+1}^{2^{m+1}} \frac{1}{2^m} P(\max_{j\le 2^{m+1}} |X_j| \ge \varepsilon 2^{mr})$$

$$\le P(\max_{j\le 2^{m+1}} |X_j| \ge \frac{\varepsilon}{2^r} 2^{(m+1)r}).$$

So (18) yields that

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\max_{j\le n} |X_j| \ge \varepsilon 2^{mr}) < \infty, \forall \varepsilon > 0,$$

$$P(\max_{j\le n} |X_j| \ge \varepsilon n^r) \to 0, n \to \infty, \forall \varepsilon > 0,$$

$$P(\max_{j\le n} |X_j| < \varepsilon n^r) \to 1, n \to \infty, \forall \varepsilon > 0.$$

Put $\varphi_0 = \lim_{n\to\infty} \varphi_n$, since $\varphi(1) < 1$, so $\varphi_0 < 1$.

$$P(\max_{j\le n} |X_j| < n^r) - \varphi(n) \to 1 - \varphi_0, n \to \infty.$$

There exists a positive integer $N \ge N_1$, such that for $n > N$,

$$P(\max_{j\le n} |X_j| < n^r) - \varphi(N) > (1-\varphi_0)/2$$

Since $\{X_n, n \ge 1\}$ is identically distributed, so it follows from (19) and (22) that

$$\infty > \sum_{n=1}^{\infty} \frac{1}{n} P(\max_{j\le n} |X_j| \ge n^r)$$

$$= \sum_{n=1}^{N} \frac{1}{n} P(\max_{j\le n} |X_j| \ge n^r)$$

$$+ \sum_{n=N+1}^{\infty} \frac{1}{n} P(\max_{j\le n} |X_j| \ge n^r)$$

$$\ge \frac{1}{N} \sum_{n=1}^{N} P(|X_n| \ge n^r)$$

$$+ \sum_{n=N+1}^{\infty} \frac{1}{n} \sum_{i=1}^{[\frac{n}{N}]} P(|X_{(i-1)N+1}| \ge n^r, \max_{iN+1<j\le n} |X_j| \ge n^r)$$

$$\ge \frac{1}{N} \sum_{n=1}^{N} P(|X_n| \ge n^r)$$

$$+ \frac{1-\varphi_0}{2} \frac{1}{N(N+1)} \sum_{n=N+1}^{\infty} P(|X_n| \ge n^r)$$

$$\ge C_1 \sum_{n=1}^{N} P(|X_n| \ge n^r)$$

$$\ge C_2 E|X_1|^{1/r}$$

Where $[x]$ denotes the integer part of x, C_1, C_2 are constants. The proof of theorem 1 is complete.

Proof of theorem 2.

Since a ρ-mixing sequence is an α-mixing sequence, $\alpha(n)/4 \le \rho(n)$, so the necessity of theorem 2 follows from that of theorem 1 immediately.

Put $\alpha = 1/\mathrm{p}, (1 \le p < 2)$ by lemma 5 we have

$$\sum_{n=1}^{\infty} n^{-1} P(\max_{j\le n} |S_j| \ge \varepsilon n^{1/p}) < \infty. \qquad (19)$$

It is easy to see (20)

$$\lim_{n\to\infty} n^{-1/p} S_n = 0. \quad a.s. \qquad (21)$$

This completes the proof of the sufficiency of theorem 2.

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